DIFFEOLOGICAL PRINCIPAL BUNDLES AND PRINCIPAL INFINITY BUNDLES

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Abstract. In this paper, we study diffeological spaces as certain kinds of discrete simplicial presheaves on the site of cartesian spaces with the coverage of good open covers. The Čech model structure on simplicial presheaves provides us with a notion of ∞-stack cohomology of a diffeological space with values in a diffeological abelian group A. We compare ∞-stack cohomology of diffeological spaces with two existing notions of Čech cohomology for diffeological spaces in the literature [KWW21] [Igl20]. Finally, we prove that for a diffeological group G, that the nerve of the category of diffeological principal G-bundles is weak homotopy equivalent to the nerve of the category of G-principal ∞-bundles on X, bridging the bundle theory of diffeology and higher topos theory.

Contents

1. Introduction

Principal G-bundles and Čech cohomology are important tools in the study of smooth manifolds. However, in recent years, the desire to expand the typical objects of study in differential geometry has led to various frameworks in which one can define a "generalized smooth space." In this paper, we draw a connection between two such frameworks. One of them is diffeology, as popularized in the textbook [Igl13]. A diffeological space consists of a set X, and a set $\mathcal{D}_X$ of functions $\mathbb{R}^n \to X$ satisfying three simple conditions. While the definition of a diffeological space is simple, a large number of interesting spaces outside the purview of classical differential geometry can be given a diffeology. Every finite dimensional smooth manifold inherits a canonical diffeology,

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as does the set $C^\infty(X,Y)$ of smooth maps between any two diffeological spaces. In fact the category of diffeological spaces is complete, cocomplete and cartesian closed. More precisely it is a quasi-topos [BH11]. This is of course not the case for the category of finite dimensional smooth manifolds, and thus provides a "better" category in which to work. Various notions of classical differential geometry have been defined for diffeological spaces, like differential forms, deRham cohomology, fiber bundles, tangent spaces [CW15], and recently Čech cohomology [Igl20] [KWW21].

The second framework is higher topos theory. Here the objects of interest are ∞-stacks over the site Cart of cartesian spaces, or more colloquially known as smooth ∞-groupoids. Many constructions of classical differential geometry can be extended to smooth ∞-groupoids, with interesting applications. One such extension is the notion of a principal ∞-bundle, as defined in [NSS14a] and [NSS14b]. Classical principal bundles and non-abelian bundle gerbes are particular examples of principal ∞-bundles. Principal ∞-bundles allow for a robust framework wherein one can study twisted, equivariant or differential refinements of generalized cohomology theories. For more on this theory we recommend the texts [FSS+12], [ADH21], [Sch13], [BNV16]. In this paper, we will use the presentation of this theory by simplicial presheaves. Thus, the reader does not need to be comfortable with ∞-categories in order to read this paper.

Our contribution is to give a connection between the aforementioned frameworks. Baez-Hoffnung proved in [BH11] that the category of diffeological spaces is equivalent to the category ConSh(Open) of concrete sheaves on Open, the category of open subsets of cartesian spaces and smooth maps, equipped with the coverage of open covers. Our first result Proposition 3.34 proves that this category is equivalent to several other categories of concrete sheaves on various sites, including the site Cart of cartesian spaces with good open covers. While this result is known to experts, and a consequence of this result appears in the literature [WW14, Lemma 2.9], Remark 2.3 explains that there are some advantages to using our Definition 2.2 over the classical definition.

The category of concrete sheaves on Cart embeds fully faithfully into the category sPre(Cart) of simplicial presheaves on Cart. A simplicial presheaf A is an ∞-stack if and only if it is fibrant in the Čech model structure on sPre(Cart). If A is an ∞-stack, whose k-fold delooping $\overline{W}^k A$ exists, then the ∞-stack cohomology of a simplicial presheaf X with coefficients in A is given by the connected components of the derived mapping space

$$H^k_\infty(X,A) = \mathbb{R}\mathrm{Hom}(X,\overline{W}^k A) = \pi_0 \underline{\mathsf{sPre(Cart)}}(QX,\overline{W}^k A),$$

where Q is a cofibrant replacement functor for the Čech model structure. This functor Q has a simple and explicit description when X is a diffeological space. Our second result, Proposition 4.20, proves that QX is the nerve of a diffeological category, whose diffeological space of objects is the coproduct $\coprod_{p\in \mathsf{Plot}(X)} U_p$ of the domains of plots of X, which also appears in [Igl13]. The diffeological space of morphisms is the coproduct $\coprod_{f_0 : U_{p_1} \to U_{p_0}} U_{p_1}$ of all domains of all morphisms in the plot category $\mathsf{Plot}(X)$ of X. This realization allows us to connect the theory of diffeological spaces to the theory of smooth ∞-groupoids.

If A is a diffeological abelian group, then we prove in Section 4.3 that A is an ∞-stack, whose k-fold delooping $\overline{W}^k A$ exist and are ∞-stacks, for every k. Our third result, Corollary 4.38, provides an explicit cochain complex that calculates the ∞-stack cohomology of a diffeological space X with values in A.

There are two other examples of diffeological categories of diffeological spaces that act as "resolutions" that already exist in the literature, namely the Čech groupoid $\check{C}(X)$ of [KWW21] and the gauge monoid $B//M$ of [Igl20]. These resolutions are used to define different notions of Čech cohomology for diffeological spaces. Our fourth result, Proposition 4.41, proves that for a diffeological space X and diffeological abelian group A, ∞-stack cohomology $H^k_\infty(X,A)$ and the Čech cohomology $H^k_{KWW}(X,A)$ of [KWW21] agree for $k = 0,1$.

Our fifth and main result of the paper, Theorem 5.20, states that for a diffeological group G, and diffeological space X, the nerve of the category $\mathsf{DiffPrin}_G(X)$ of diffeological principal G-bundles on X is weak homotopy equivalent to the nerve of the category of G-principal ∞-bundles on X. Thus we provide a bridge between the bundle theories of the two frameworks for higher differential geometry as discussed above.

The paper is organized as follows. In Section 2, we will give some background information about diffeological spaces. Note that Definition 2.2 of a diffeological space is not standard. In Section 3, we will show that the category of diffeological spaces under Definition 2.2 is equivalent to the category of concrete sheaves on the category of cartesian spaces equipped with the coverage of good open covers. We also compare several categories of concrete sheaves on various sites, and show that they are all equivalent. In Section 4, we will review the Čech model structure on simplicial presheaves over cartesian spaces. Proposition 4.20 provides a cofibrant replacement of a diffeological space as the nerve of a diffeological category. We compare this diffeological category to two other diffeological categories $\check{C}(X)$ and $B//M$, which have been introduced in [KWW21] and [Igl20], respectively. From these three diffeological categories, we obtain three separate notions of Čech cohomology for diffeological spaces, and compare them in Section 4.3. In Section 5, we turn to the main result of this paper, that if G is a diffeological group and X is a diffeological space, then the nerve of the category of principal G-bundles on X is weak homotopy equivalent to the nerve of the category of G-principal ∞-bundles over X.

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2. Diffeological Spaces

In this section we give some background on diffeological spaces, all of which can be found in [Igl13], and then focus on the perspective of diffeological spaces as concrete sheaves on Cart.

Definition 2.1. Let M be a finite dimensional smooth manifold[1]. We say a collection of subsets $\mathcal{U} = \{U_i \subseteq M\}_{i \in I}$ is an **open cover** if each U_i is an open subset of M, and

[1] We will assume throughout this paper that manifolds are Hausdorff and paracompact.

$\bigcup_{i\in I} U_i = M$. If U is a finite dimensional smooth manifold diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, we call U a **cartesian space**. We call $\mathcal{U} = \{U_i \subseteq M\}$ a **cartesian open cover** of a manifold M if it is an open cover of M and every U_i is a cartesian space. We say that $\mathcal{U}$ is a **good open cover** if it is a cartesian open cover, and further every finite non-empty intersection $U_{i_0\dots i_k} = U_{i_0} \cap \dots \cap U_{i_k}$ is a cartesian space.

Let Man denote the category whose objects are finite dimensional smooth manifolds and whose morphisms are smooth maps. Let Cart denote the full subcategory whose objects are cartesian spaces. Given a set X, let $\mathsf{Param}(X)$ denote the set of **parametrizations** of X, namely the collection of set functions $p : U \to X$, where $U \in \mathsf{Cart}$.

Definition 2.2. A **diffeology** on a set X, consists of a collection $\mathcal{D}$ of parametrizations $p : U \to X$ satisfying the following three axioms:

(1) $\mathcal{D}$ contains all points $\mathbb{R}^0 \to X$,
(2) If $p : U \to X$ belongs to $\mathcal{D}$, and $f : V \to U$ is a smooth map, then $pf : V \to X$ belongs to $\mathcal{D}$, and
(3) If $\{U_i \subseteq U\}_{i\in I}$ is a good open cover of a cartesian space U, and $p : U \to X$ is a parametrization such that $p|_{U_i} : U_i \to X$ belongs to $\mathcal{D}$ for every $i \in I$, then $p \in \mathcal{D}$.

A set X equipped with a diffeology $\mathcal{D}$ is called a **diffeological space**. Parametrizations that belong to a diffeology are called **plots**. We say a set function $f : X \to Y$ between diffeological spaces is **smooth** if for every plot $p : U \to X$ in $\mathcal{D}_X$, the composition $pf : U \to Y$ belongs to $\mathcal{D}_Y$.

Denote the category whose objects are diffeological spaces and morphisms are smooth maps between them by Diff. An isomorphism in this category is called a **diffeomorphism**.

Remark 2.3. Note that Definition 2.2 is not the exact definition of diffeological spaces as usually given in the literature, such as [Igl13, Article 1.5]. However it is precisely the definition of diffeological space as defined in [Pav22, Definition 2.7] and [SS21, Notation 3.3.15], as we will prove in Theorem 3.16. We will call these **classical diffeological spaces** and denote their category by Diff'.

In Section 3, leveraging [BH11], we will explain how to think of diffeological spaces as concrete sheaves on the site $(\mathsf{Cart}, j_{\text{good}})$, namely cartesian spaces with the coverage of good open covers. Leveraging this perspective we show that Diff is equivalent to Diff'.

However there are real advantages to using Diff over Diff', one of them being Lemma 5.5, which is false for Diff'. There are other more technical advantages as well. In Section 4, we will consider Diff embedded into the category $\mathsf{sPre}(\mathsf{Cart})$, which can be given the Čech model structure $\check{\mathsf{sPre}}(\mathsf{Cart})$. If one uses j_{open} instead of j_{good} on Cart, and $\mathcal{U}$ is an arbitrary cartesian open cover of a cartesian space U, then there is no guarantee that $\check{C}(\mathcal{U})$ will be projective cofibrant. Using good open covers ensures that it is projective cofibrant, which is necessary for much of the theory to be developed. Using j_{good} also allows us to leverage Theorem 4.17 and Theorem 4.32, which are vital to our results.

There are other definitions for diffeological spaces that one could choose. We will document these alternative definitions and show their resulting categories are all equivalent in Section 3.

Example 2.4. If M is a finite dimensional smooth manifold, then the set of parametrizations $p : U \to M$ that are smooth in the sense of classical differential geometry forms a diffeology [Igl13, Article 4.3]. Further a map $f : M \to N$ of smooth manifolds is smooth in the classical sense if and only if it is smooth as a map of diffeological spaces. This implies there is a fully faithful embedding $\mathsf{Man} \hookrightarrow \mathsf{Diff}$.

Let $\mathcal{D}(X)$ denote the poset of all diffeologies on X ordered by inclusion. Let $\mathcal{D}_X, \mathcal{D}'_X$ be two diffeologies on X. We say that $\mathcal{D}_X$ is **coarser** than $\mathcal{D}'_X$ if $\mathcal{D}_X \subseteq \mathcal{D}'_X$, in which case we also say that $\mathcal{D}'_X$ is **finer** than $\mathcal{D}_X$.

Definition 2.5. Given a diffeological space $(X, \mathcal{D}_X)$, a **generating family** $\mathcal{G}$ for $\mathcal{D}_X$ is a collection $\{V_i \xrightarrow{q_i} X\}$ of parametrizations such that for every plot $U \xrightarrow{p} X$ there exists a good open cover $\{U_i \hookrightarrow U\}$ of U and a smooth map $f_i : U_i \to V_i$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "p"] & X \\ U_i \arrow[u, hook] \arrow[r, "f_i"'] & V_i \arrow[u, "q_i"'] \end{tikzcd} \tag{1}$$

If $\mathcal{G}$ is a generating family for $\mathcal{D}_X$, we write $\mathcal{D}_X = \langle \mathcal{G} \rangle$. Conversely, given any set of parametrizations $\mathcal{G} \subseteq \mathsf{Param}(X)$, such that the images of the parametrizations in $\mathcal{G}$ cover X, then let $\langle \mathcal{G} \rangle$ denote the diffeology it generates, namely plots of $\langle \mathcal{G} \rangle$ are plots p that satisfy (1).

Example 2.6. The diffeology on a set X generated by the empty set $\langle \varnothing \rangle$ is precisely the discrete diffeology $\mathcal{D}^\circ_X$, since it is the finest possible diffeology on X. Similarly $\langle \mathsf{Param}(X) \rangle = \mathcal{D}^\bullet_X$ is the codiscrete diffeology.

Definition 2.7. Given a set X, a diffeological space $(Y, \mathcal{D}_Y)$ and a set function $X \xrightarrow{f} Y$, there exists a diffeology:

$$f^*(\mathcal{D}_Y) = \{p \in \mathsf{Param}(X) : fp \in \mathcal{D}_Y\} \tag{2}$$

called the **pullback diffeology** on X.

Similarly if $(X, \mathcal{D}_X)$ is a diffeological space, Y is a set, and $X \xrightarrow{f} Y$ is a set function, then there exists a diffeology:

$$f_*(\mathcal{D}_X) = \langle f \circ \mathcal{D}_X \rangle \tag{3}$$

called the **pushforward diffeology** on Y, and where

$$f \circ \mathcal{D}_X = \{V \xrightarrow{p} Y \in \mathsf{Param}(Y) : p = f\varphi, \text{ where } \varphi \in \mathcal{D}_X\}.$$

This implies that a parametrization $U \xrightarrow{p} Y$ is a plot of $f_*(\mathcal{D}_X)$ if it is constant or there exists a good open cover $\{U_i \subseteq U\}$ and plots $p_i : U_i \to X$ such that the following diagram commutes.

$$\begin{tikzcd} U_i \arrow[r, hook] \arrow[d, "p_i"'] & U \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{4}$$

Definition 2.8. A map $(X,\mathcal{D}_X) \xrightarrow{f} (Y,\mathcal{D}_Y)$ between diffeological spaces is called an:

- **induction** if it is injective and $f^*(\mathcal{D}_Y) = \mathcal{D}_X$
- **subduction** if it is surjective and $f_*(\mathcal{D}_X) = \mathcal{D}_Y$.

Definition 2.9. Let $(X,\mathcal{D}_X)$ be a diffeological space and $A \overset{\iota}{\hookrightarrow} X$ a subset. Then consider:

$$\mathcal{D}_A^X = \iota^*(\mathcal{D}_X) = \{U \xrightarrow{p} A : \iota p \in \mathcal{D}_X\} \tag{5}$$

the pullback diffeology on A by the inclusion map. We call this the **subspace diffeology** on A. ι is always an induction when A is equipped with the subspace diffeology.

Definition 2.10. Let $(X,\mathcal{D}_X)$ be a diffeological space and $X \xrightarrow{\pi} X/\sim$ the projection map onto the quotient by some equivalence relation taking a point $x \in X$ to its equivalence class $[x] \in X/\sim$. Then consider:

$$D_\sim^X = \pi_*(\mathcal{D}_X) \tag{6}$$

the pushforward diffeology on $X/\sim$ by the projection map. We call this the **quotient diffeology**. π is always a subduction when $X/\sim$ is equipped with the quotient diffeology.

A parametrization $U \xrightarrow{p} X/\sim$ is a plot if it is locally a composition of π and a plot in X. In other words, there exists a covering $\{U_i \hookrightarrow U\}$, and plots $U_i \to X$ for every i, such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "p"] & X/\sim \\ U_i \arrow[u, hook] \arrow[r, "p_i"'] & X \arrow[u, "\pi"'] \end{tikzcd}$$

In fact, the category of diffeological spaces is complete and cocomplete. Suppose $F : \mathsf{J} \to \mathsf{Diff}$ is a diagram of diffeological spaces. Then a parametrization $p : U \to \lim F$, (where we are taking $\lim F$ to be the limit of the underlying sets of the F_j) is a plot if and only if the composite $U \xrightarrow{p} \lim F \to F_j$ is a plot for every $j \in \mathsf{J}$. Similarly a parametrization $p : \mathsf{J} \to \operatorname{colim} F$ is a plot if and only if there exists a good open cover $\{U_i \to U\}$ and plots $U_i \xrightarrow{p_i} F_{j_i}$ for each i, such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "p"] & \operatorname{colim} F \\ U_i \arrow[u, hook] \arrow[r, "p_i"'] & F_{j_i} \arrow[u] \end{tikzcd}$$

Definition 2.11. Given any two diffeological spaces X, Y, consider the set $\mathsf{Diff}(X,Y)$ of smooth maps $X \xrightarrow{f} Y$. Equip it with the **functional diffeology**:

$$\mathcal{D}_{X\to Y} = \{U \xrightarrow{p} \mathsf{Diff}(X,Y) : U \times X \xrightarrow{p^\#} Y \text{ is smooth}\} \tag{7}$$

where $p^\# : U \times X \to Y$ is the transpose map, defined by $p^\#(u,x) = p(u)(x)$. Denote the diffeological space by $C^\infty(X,Y) := (\mathsf{Diff}(X,Y), \mathcal{D}_{X\to Y})$ or for shorthand by Y^X.

The functional diffeology makes Diff a Cartesian closed category. We will see in Section 3 that Diff is in fact a quasitopos.

3. Diffeological Spaces as Concrete Sheaves

A major development in the theory of diffeological spaces was made in [BH11], which showed that the category of diffeological spaces can be identified with concrete sheaves on the site of open subsets of $\mathbb{R}^n$ for all $n \in \mathbb{N}$. Here we introduce the theory necessary to understand this result. In Section 3.1, we will compare this definition with concrete sheaves on Cart and Man with several coverages. Everything in this section is well known to experts in topos theory, however it may not be well known to diffeologists, for whom this material has real effects on their theory.

Definition 3.1. Let $\mathcal{C}$ be a category, and $U \in \mathcal{C}$. A **family of morphisms** over U is a set of morphisms $r = \{r_i : U_i \to U\}_{i\in I}$ in $\mathcal{C}$ with codomain U.

A **refinement** of a family of morphisms $t = \{t_j : V_j \to U\}_{j\in J}$ over U consists of a family of morphisms $r = \{r_i : U_i \to U\}_{i\in I}$, a function $\alpha : I \to J$ and for each $i \in I$ a map $f_i : U_i \to V_{\alpha(i)}$ making the following diagram commute:

$$\begin{tikzcd} U_i \arrow[rr, "f_i"] \arrow[dr, "r_i"'] & & V_{\alpha(i)} \arrow[dl, "t_{\alpha(i)}"] \\ & U & \end{tikzcd} \tag{8}$$

If r is a refinement of t, with maps $f_i : U_i \to V_{\alpha(i)}$, then we write $f : r \to t$.

We wish to consider added structure to a category that generalizes the notion of a topology. We will use families of morphisms as a generalized notion of "open cover."

Definition 3.2. A **collection of families** j on a category $\mathcal{C}$ consists of a set $j(U)$ for each $U \in \mathcal{C}$, whose elements $\{r_i : U_i \to U\} \in j(U)$ are families of morphisms over U.

We call a collection of families j on $\mathcal{C}$ a **coverage** if it satisfies the following property: for every $\{r_i : U_i \to U\} \in j(U)$, and every map $g : V \to U$ in $\mathcal{C}$, then there exists a family $\{t_j : V_j \to V\} \in j(V)$ such that gt_j factors through some r_i. Namely for every t_j there exists some i and some map $s_j : V_j \to U_i$ making the following diagram commute:

$$\begin{tikzcd} V_j \arrow[r, "s_j"] \arrow[d, "t_j"'] & U_i \arrow[d, "r_i"] \\ V \arrow[r, "g"'] & U \end{tikzcd} \tag{9}$$

The families $\{r_i : U_i \to U\} \in j(U)$ are called **covering families** over U. If a map $r_i : U_i \to U$ belongs to a covering family $r \in j(U)$, then we say that r_i is a **covering map**.

If $\mathcal{C}$ is a category, and j is a coverage on $\mathcal{C}$, then we call the pair $(\mathcal{C}, j)$ a **site**.

Example 3.3. Let X be a topological space and let $\mathcal{O}(X)$ denote the partially ordered set of open subsets of X. Let j_X denote the collection of familes on $\mathcal{O}(X)$ such that $j_X(U)$ is the set of all open covers of U, namely $\{U_i \subseteq U\} \in j_X(U)$ if $\bigcup_i U_i = U$.

This collection of families is a coverage, for suppose we have fixed an open cover $\{U_i \subseteq U\}$ and an open subset $V \subseteq U$. Then $\{V \cap U_i \subseteq V\}$ is an open cover of V, and $V \cap U_i \subseteq U_i$. We call j_X the **open cover coverage** of X.

Example 3.4. Define a collection of families j_{open} on Man as follows: For $M \in \mathsf{Man}$, let $j_{\text{open}}(M)$ denote the collection of open covers of M. Then j_{open} is a coverage. Indeed if $\{U_i \subseteq M\}$ is an open cover and $f : N \to M$ is a smooth map, then $\{f^{-1}(U_i) \subseteq N\}$ is an open cover of N satisfying (9).

Now consider the following full subcategories

$$\mathsf{Cart} \hookrightarrow \mathsf{Open} \hookrightarrow \mathsf{Man}.$$

Where Cart is the full subcategory whose objects are cartesian spaces and Open is the full subcategory whose objects are diffeomorphic to open subsets of a cartesian space. The collection of families j_{open} can be restricted to Open and is clearly a coverage there as well.

Notice however that if we restrict j_{open} to Cart, and U is a cartesian space, then an open cover $\{U_i \subseteq U\}$ is a covering family for j_{open} if and only if it is a cartesian open cover, otherwise it could not be a collection of morphisms in Cart. For Man and Open any open cover will do. However if $\{U_i \subseteq U\}$ is a cartesian open cover and $f : V \to U$ is a smooth map, there is no reason that $\{f^{-1}(U_i) \subseteq V\}$ will be a cartesian open cover. However as we will see in Example 3.5, every open cover can be refined by a cartesian open cover, and thus j_{open} is indeed a coverage on Cart.

Example 3.5. Define a collection of families j_{good} on Man as follows: For $M \in \mathsf{Man}$, let $j_{\text{good}}(M)$ denote the collection of good open covers as in Definition 2.1 of M. Let us show that the good covers form a coverage. If $\{U_i \subseteq M\}$ is a good cover and $g : N \to M$ a smooth map, then $\{g^{-1}(U_i) \subseteq N\}$ is an open cover, but not necessarily good. By [BT+82, Corollary 5.2], this open cover can be refined by a good open cover $\{W_k \subseteq N\}$ so that for every W_k in the good open cover, there exists a U_i such that $W_k \subseteq g^{-1}(U_i)$, and thus the following diagram commutes:

$$\begin{tikzcd} W_k \arrow[r, hook] \arrow[d, hook] & g^{-1}(U_i) \arrow[r, "g|_{g^{-1}(U_i)}"] & U_i \arrow[d, hook] \\ N \arrow[rr, "g"'] & & M \end{tikzcd}$$

Thus j_{good} is a coverage on Man. Similarly it defines a coverage on Cart and Open.

Definition 3.6. Let Smooth denote a site of the form $(\mathcal{C}, j)$ with $\mathcal{C} \in \{\mathsf{Cart}, \mathsf{Open}, \mathsf{Man}\}$ and $j \in \{j_{\text{open}}, j_{\text{good}}\}$. We will call any such site a **smooth site**.

Example 3.7. We note here that the collection of families j_{sub} of subductions on the category Diff of diffeological spaces is a coverage, because the pullback of a subduction is a subduction. We will not use this observation in this section, but it will come up in Section 4.2 when we talk about diffeological categories.

Coverages are those collections of families with the least amount of structure with which we can define sheaves on $\mathcal{C}$.

Definition 3.8. A **presheaf** on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathsf{Set}$. An element $x \in F(U)$ for an object $U \in \mathcal{C}$ is called a **section** over U. If $f : U \to V$ is a map in $\mathcal{C}$, and $x \in F(V)$, then we sometimes denote $F(f)(x)$ by $x|_U$. If $\{r_i : U_i \to U\}_{i \in I}$ is a covering family, then a **matching family** is a collection $\{x_i\}_{i\in I}$, $x_i \in F(U_i)$, such that given a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "f"'] & U_j \arrow[d, "r_j"] \\ U_i \arrow[r, "r_i"'] & U \end{tikzcd}$$

then $F(f)(x_i) = F(g)(x_j)$ for all $i, j \in I$. An **amalgamation** x for a matching family $\{x_i\}$ is a section $x \in F(U)$ such that $x_i|_U = x$ for all i.

Definition 3.9. Given a family of morphisms $r = \{r_i : U_i \to U\}$ in a category $\mathcal{C}$, we say that a presheaf $F : \mathcal{C}^{op} \to \mathsf{Set}$ is a **sheaf on** r if every matching family $\{s_i\}$ of F over r has a unique amalgamation. If j is a coverage on a category $\mathcal{C}$, we call F a **sheaf** on $(\mathcal{C}, j)$ if it is a sheaf on every covering family of j. Let $\mathsf{Sh}(\mathcal{C}, j)$ denote the full subcategory of $\mathsf{Pre}(\mathcal{C})$ whose objects are sheaves on $(\mathcal{C}, j)$.

Remark 3.10. If $(\mathcal{C}, j)$ is a site that has pullbacks, then we can equivalently express the condition for F being a sheaf as requiring that for every $U \in \mathcal{C}$ and every covering family $\{U_i \to U\} \in j(U)$, the diagram:

$$F(U) \longrightarrow \textstyle\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \tag{10}$$

is an equalizer. This is how the sheaf condition is often presented in the literature.

Example 3.11. Given a smooth manifold M, the presheaf

$$U \mapsto C^\infty(U, M)$$

which we denote by either $\underline{M}$ or just M, is a sheaf on Smooth. Another important example of a sheaf is

$$U \mapsto \Omega^k(U)$$

for any $k \geq 0$. If V is a cartesian space, we will denote its image under the Yoneda embedding by yV. This is the presheaf

$$U \mapsto C^\infty(U, V)$$

and as above is a sheaf. We call a site $(\mathcal{C}, j)$ where every representable presheaf is a sheaf **subcanonical**. It is not hard to see that Smooth is a subcanonical site.

We wish to single out those sheaves that are in some sense a set with extra structure.

Definition 3.12. A site $(\mathcal{C}, j)$ is **concrete** if:

(1) it is subcanonical,
(2) it has a terminal object $*$,
(3) the functor $\mathcal{C}(*, -) : \mathcal{C} \to \mathsf{Set}$ is faithful, and
(4) for every covering family $\{U_i \to U\}$, the family of maps $\mathcal{C}(*, U_i) \to \mathcal{C}(*, U)$ is **jointly surjective**, namely the map $\coprod_i \mathcal{C}(*, U_i) \to \mathcal{C}(*, U)$ is surjective.

It is not hard to show that all of the smooth sites are concrete.

Definition 3.13. If $(\mathcal{C}, j)$ is a concrete site and F is a presheaf, then we call $F(*)$ its **underlying set**, and for any $U \in \mathcal{C}$ there always exists a map

$$\phi_U : F(U) \to \mathsf{Set}(\mathcal{C}(*, U), F(*))$$

defined by $\phi_U(x) = (u \mapsto F(u)(x))$. We say a sheaf F is **concrete** if for every object $U \in \mathcal{C}$, the function ϕ_U is injective. Let $\mathsf{ConSh}(\mathcal{C}, j)$ denote the full subcategory of concrete sheaves on a concrete site $(\mathcal{C}, j)$.

Example 3.14. For any smooth manifold M, the sheaf $\underline{M}$ on Smooth is concrete. This is equivalent to saying that for every $U \in \mathsf{Smooth}$ the function

$$\phi_U : C^\infty(U, M) \to \mathsf{Set}(U(*), M(*))$$

is injective, which is the same thing as saying that the set of smooth maps from U to M is a subset of all set functions from U to M.

Note that the sheaf Ω^k is not concrete on Smooth. Indeed if $U \in \mathsf{Smooth}$, then ϕ_U takes the form

$$\phi_U : \Omega^k(U) \to \mathsf{Set}(U(*), \Omega^k(*))$$

but $\Omega^k(*) = \{0\}$ is the zero vector space, thus $\mathsf{Set}(U(*), \{0\}) = *$ is the singleton set. It is well known that unless $U = *$, $\Omega^k(U)$ is not zero dimensional for any k.

Theorem 3.15 ([BH11, Proposition 24])**.** The category Diff' of classical diffeological spaces is equivalent to the category of concrete sheaves on Open with the open cover coverage,

$$\mathsf{Diff}' \simeq \mathsf{ConSh}(\mathsf{Open}, j_{\mathrm{open}}).$$

However, the proof of [BH11, Proposition 24] can be applied nearly word for word to prove the following.

Theorem 3.16. The category Diff of diffeological spaces as defined in Definition 2.2 is equivalent to the category of concrete sheaves on Cart with the good open cover coverage

$$\mathsf{Diff} \simeq \mathsf{ConSh}(\mathsf{Cart}, j_{\mathrm{good}}).$$

Thus in Section 3.1 we will prove $\mathsf{Diff} \simeq \mathsf{Diff}'$ by showing that $\mathsf{ConSh}(\mathsf{Cart}, j_{\mathrm{good}}) \simeq \mathsf{ConSh}(\mathsf{Open}, j_{\mathrm{open}})$.

Theorem 3.16 allows us to make a perspective shift. Constructions made in Diff can be compared with already defined notions of sheaves. For example a differential k-form ω on a diffeological space X [Igl13, Article 6.28], is precisely a map

$$X \xrightarrow{\omega} \Omega^k$$

of sheaves on Cart. This viewpoint on diffeological spaces, namely as concrete sheaves on Cart, will also be the starting point for Section 4, where we consider the fully faithful embedding of presheaves into simplicial presheaves. Since concrete sheaves are in particular presheaves, this means that there is a fully faithful embedding of diffeological spaces into simplicial presheaves, where we have a powerful homotopy theory to leverage.

3.1. **Comparison of Site Structures.** Here we will prove that the definition of diffeological spaces as given in Definition 2.2 is equivalent to that usually presented in the literature, such as [Igl13, Article 1.5], in the sense that their categories are equivalent. Further we will show other possible alternative definitions that have not appeared in the literature have equivalent categories as well. An example of this is [WW14, Lemma 2.9]. The results of this section include this result.

We will do this by exploiting Theorem 3.16, and studying concrete sheaves over the smooth sites. Now coverages are those collections of families with the least amount of structure with which we can define sheaves on $\mathcal{C}$. There could be many different coverages which give rise to equivalent categories of sheaves. It can therefore be difficult to see directly when coverages give rise to the same sheaves. We will define a more restricted kind of coverage, known as a Grothendieck coverage or Grothendieck topology, which will make such comparison easier.

Definition 3.17. A **sieve** R is a family of morphisms that is closed under precomposition, namely if $V \xrightarrow{g} U_i$ is a map in $\mathcal{C}$, and $U_i \xrightarrow{r_i} X \in R$, then $V \xrightarrow{r_i g} X \in R$.

Given a category $\mathcal{C}$ and an object $U \in \mathcal{C}$, there is a bijection between sieves on X and subfunctors $R \hookrightarrow yU$, where $yU = (V \mapsto \mathcal{C}(V,U))$ denotes the Yoneda embedding on U. Indeed, given a sieve R, we can define a subfunctor $\widetilde{R} \hookrightarrow yU$ by setting $\widetilde{R}(V) = \{f : V \to U : f \in R\}$ and noting that being a sieve implies that $\widetilde{R}$ is functorial under precomposition, and conversely if $\widetilde{R} \hookrightarrow yU$ is a subfunctor, then we can define a sieve R by setting $R = \bigcup_{V \in \mathcal{C}} \widetilde{R}(V)$. Thus for the rest of this section a sieve will mean both a kind of family of morphisms and a subfunctor of the Yoneda embedding. If $U \in \mathcal{C}$ is an object, then we call yU the **maximal sieve**. This is equivalently the family of all morphisms with codomain U.

For any family of morphisms $r = \{r_i : U_i \to U\}$ over U, we can construct the smallest sieve $R = \overline{r}$ containing it as follows. Let R be the set of morphisms $f : V \to U$ such that f factors as:

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "g"'] & & U \\ & U_i \arrow[ur, "r_i"'] & \end{tikzcd}$$

where $r_i : U_i \to U \in r$, and g is a morphism in $\mathcal{C}$. In this case we say that r **generates** the sieve R.

Lemma 3.18 ([Joh02, C2.1 Lemma 2.1.3])**.** Suppose that j is a coverage on a category $\mathcal{C}$. Then a presheaf F is a sheaf on a family of morphisms $r = \{U_i \to U\}$ if and only if it is a sheaf on the sieve $R = \overline{r}$ it generates.

Definition 3.19. We say that a collection of families j is **sifted** if every $r \in j(U)$ is a sieve. If j is further a coverage, we call it a sifted coverage. We call covering families of sifted coverages **covering sieves**.

Lemma 3.20. Let R be a sieve over an object U in a category $\mathcal{C}$ and F a presheaf on $\mathcal{C}$. A collection $\{s_f \in F(V)\}_{f \in R}$ of sections for every $f : V \to U$ in R is a matching family iff $F(g)(s_f) = s_{fg}$ for every morphism $g : W \to V$ in $\mathcal{C}$.

Proof. ($\Rightarrow$) Suppose $\{s_f\}$ is a matching family, then consider the commutative diagram:

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d, "g"'] & W \arrow[d, "gf"] \\ V \arrow[r, "f"'] & U \end{tikzcd}$$

this implies that $F(g)(s_f) = s_{fg}$.

($\Leftarrow$) Suppose we have a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & V' \arrow[d, "f'"] \\ V \arrow[r, "f"'] & U \end{tikzcd}$$

where $f, f' \in R$. Then $F(g)(s_f) = s_{fg} = s_{f'h} = F(h)(s_{f'})$, thus $\{s_f\}$ is a matching family. $\square$

If j is a coverage, then let $\overline{j}$ denote the collection of families where $R \in \overline{j}(U)$ if $R = \overline{r}$ for some $r \in j(U)$. We call $\overline{j}$ the **sifted closure** of j.

Lemma 3.21. The collection of families $\overline{j}$ is a sifted coverage of $\mathcal{C}$.

Proof. Clearly $\overline{j}$ is sifted. We wish to show it is a coverage. Suppose we have a covering family $R \in \overline{j}(U)$, and a map $g : V \to U$. We wish to show that there is a covering family $R' \in \overline{j}(V)$ such that for every map $k \in R'$, gk factors through some $l \in R$. Since $R = \overline{r}$, we know that since j is a coverage, there exists some covering family $t \in j(V)$ with the corresponding property. In other words, for every map $k \in R'$ there is a commutative diagram:

$$\begin{tikzcd} W \arrow[d, "k_j"] \arrow[dd, bend right=60, "k"'] & \\ V_j \arrow[r, "s_j"] \arrow[d, "t_j"] & U_i \arrow[d, "r_i"] \\ V \arrow[r, "g"'] & U \end{tikzcd}$$

but then $l = r_i s_j k_j$ is a morphism in $R = \overline{r}$. Thus gk factors through l as it is equal to it. $\square$

Corollary 3.22. Given a coverage j on a category $\mathcal{C}$, a presheaf F is a sheaf on $(\mathcal{C}, j)$ if and only if it is a sheaf on $(\mathcal{C}, \overline{j})$. In other words $\mathsf{Sh}(\mathcal{C}, j) = \mathsf{Sh}(\mathcal{C}, \overline{j})$.

Proof. This follows from Lemma 3.18 and Lemma 3.21. □

Now if $R \hookrightarrow yU$ is a sieve, and $f : V \to U$ is a morphism in $\mathcal{C}$, then let f^*R denote the set of morphisms $g : W \to V$ such that $fg \in R$. This is equivalently the subfunctor $f^*R \hookrightarrow yV$ given by the pullback in $\mathrm{Pre}(\mathcal{C})$

$$\begin{tikzcd} f^*R \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, hook] \\ yV \arrow[r, "f"'] & yU \end{tikzcd}$$

Definition 3.23. A **Grothendieck coverage** is a sifted collection of families J on a category $\mathcal{C}$ satisfying the following conditions:

(C) J is a coverage,
(M) for any object $U \in \mathcal{C}$, the maximal sieve $yU \in J(U)$, and
(L) if $R \in J(U)$ and S is another sieve on U such that for each $f : V \to U \in R$, the sieve $f^*(S)$ belongs to $J(V)$.

If $(\mathcal{C}, J)$ is a Grothendieck coverage, then we call its sieves $R \in J(U)$ **covering sieves**.

Remark 3.24. Grothendieck coverages are usually referred to as Grothendieck topologies in the literature, but are typically presented with the following condition (C'): If $R \in J(U)$ and $f : V \to U$ any morphism in $\mathcal{C}$, then $f^*R \in J(V)$, instead of the condition (C). It is not hard to show that these are equivalent definitions, see [Joh02, C2.1 Page 541].

Lemma 3.25. Let $\mathcal{C}$ be a category and $R \hookrightarrow yU$ a sieve. If $g : V \to U$ is a map in R, then $g^*R = yV$.

Proof. If R is a sieve on U, then $g^*R = \{f : W \to V \,:\, gf \in R\}$. But R is a sieve and $g \in R$, so every map $f : W \to V$ has this property, since R is closed under precomposition. □

Lemma 3.26. Let $(\mathcal{C}, J)$ be a site with a Grothendieck coverage. Then if R, R' are sieves on U, $R \subseteq R'$ and R is a covering sieve, then R' is a covering sieve.

Proof. Let $g : V \to U \in R \subseteq R'$, then by Lemma 3.25, we know that $g^*R = g^*R' = yV$, which is a covering sieve of V by (M). Since this is true for all $g \in R$, R' is a covering sieve by (L). □

Given a set $\{J_\alpha \,:\, \alpha \in A\}$ of Grothendieck coverages, it is not hard to check that the collection of families $J := \bigcap_{\alpha \in A} J_\alpha$ defined by $J(U) = \bigcap_{\alpha \in A} J_\alpha(U)$ is a Grothendieck coverage. Thus if j is a coverage on $\mathcal{C}$, we can consider $\overline{j}$, its sifted closure. By Lemma 3.18, we can then take the intersection of the set of Grothendieck coverages that contain all of the covering sieves of $\overline{j}$, which we denote by $\tau(j)$. This will be the smallest Grothendieck coverage containing j and we will call it the Grothendieck coverage generated by j.

Lemma 3.27 ([Joh02, C2.1 Proposition 2.1.9])**.** Given a site $(\mathcal{C}, j)$, a presheaf F will be a sheaf on $(\mathcal{C}, j)$ if and only if it is a sheaf on $(\mathcal{C}, \tau(j))$.

Now we are in a position to compare different coverages on the same category. Suppose that j, j' are coverages on a category $\mathcal{C}$ such that if r' is a covering family in $j'(U)$, then there exists a covering family $r \in j(U)$ and a refinement $f : r \to r'$. We will say that j' is **subordinate** to j and write $j' \leq j$.

Proposition 3.28. Suppose that j, j' are coverages on a category $\mathcal{C}$ such that $j \leq j'$ and $j' \leq j$, then $\mathsf{Sh}(\mathcal{C}, j) = \mathsf{Sh}(\mathcal{C}, j')$.

Proof. Suppose that $j' \leq j$. Then every covering family $r' \in j'(U)$ can be refined by a covering family $r \in j(U)$. Therefore $\overline{r} \subseteq \overline{r'}$, since sieves are closed under precomposition. Now note that $\overline{r} \in \tau(j)(U)$, and thus by Lemma 3.26, $\overline{r'} \in \tau(j)(U)$. Thus if r' is a covering family of j', then $\overline{r'}$ is a covering sieve of $\tau(j)$. Thus if F is a sheaf on j, then by Lemma 3.27, it will be a sheaf on $\tau(j)$, so it will then be a sheaf on $\overline{r'}$, and thus by Lemma 3.18 it will be a sheaf on r'. Since r' was arbitrary, F is therefore a sheaf on all of j'. Thus if F is a sheaf on $(\mathcal{C}, j)$, then it will be a sheaf on $(\mathcal{C}, j')$. Conversely $j \leq j'$ proves that $\mathsf{Sh}(\mathcal{C}, j) = \mathsf{Sh}(\mathcal{C}, j')$. □

Proposition 3.29. Let j_{good} denote the good cover coverage on Man defined in Example 3.5, and j_{open} denote the open cover coverage on Man defined in Example 3.4. Then $\mathsf{Sh}(\mathsf{Man}, j_{\text{good}}) = \mathsf{Sh}(\mathsf{Man}, j_{\text{open}})$. This similarly holds for Cart and Open.

Proof. By [BT+82, Corollary 5.2], we have that $j_{\text{open}} \leq j_{\text{good}}$. Now $j_{\text{good}} \leq j_{\text{open}}$, since every good open cover is in particular an open cover. □

Corollary 3.30. The categories of concrete sheaves on Man with the open and good open coverages agree:

$$\mathsf{ConSh}(\mathsf{Man}, j_{\text{good}}) = \mathsf{ConSh}(\mathsf{Man}, j_{\text{open}}).$$

This result remains true if we replace Man with Open or Cart.

Now we wish to compare sites whose underlying categories differ. Let $\mathcal{C}$ be a category and $\mathcal{C}' \hookrightarrow \mathcal{C}$ a full subcategory. Then a sieve $R \hookrightarrow yU$ on $\mathcal{C}$ is said to be a $\mathcal{C}'$-sieve if it is generated by a family of morphisms all of whose domains are objects in $\mathcal{C}'$.

Definition 3.31. Let $(\mathcal{C}, J)$ be a category with a Grothendieck coverage, and $\mathcal{C}' \hookrightarrow \mathcal{C}$ a full subcategory. We say that $\mathcal{C}'$ is ***J*-dense** in $\mathcal{C}$ if every object $U \in \mathcal{C}$ has a covering sieve $R \in J(U)$ that is a $\mathcal{C}'$-sieve.

If $(\mathcal{C}, j)$ is a site where j is not necessarily a Grothendieck coverage, then we say that a full subcategory $\mathcal{C}' \hookrightarrow \mathcal{C}$ is j-dense if it is $\tau(j)$-dense in $(\mathcal{C}, \tau(j))$.

By [BT+82, Theorem 5.1], every finite dimensional smooth manifold has a good open cover. Thus if $\mathcal{U} = \{U_i \subseteq M\}$ denotes a good open cover of M, then $\overline{\mathcal{U}}$ is a covering sieve of $(\mathsf{Man}, \tau(j_{\text{open}}))$ and it is a Cart-sieve. Since this is true for any manifold M, it follows that Cart is j_{open}-dense in $(\mathsf{Man}, j_{\text{open}})$. By the same argument Cart is also dense in $(\mathsf{Man}, j_{\text{good}})$. This also implies that Open is dense in (Man, j) for $j \in \{j_{\text{open}}, j_{\text{good}}\}$.

Now suppose $(\mathcal{C}, J)$ is a site with a Grothendieck coverage. If $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full subcategory, define a collection of families J' on $\mathcal{C}'$ by defining $J'(U)$ to be the collection of those covering sieves $R \in J(U)$ that are also $\mathcal{C}'$-sieves. It is not hard to show that this is also a Grothendieck coverage, called the **induced coverage** on $\mathcal{C}'$, and denoted $J|_{\mathcal{C}'}$.

The following result is well-known in the literature as the **Comparison Lemma**.

Theorem 3.32 ([Joh02, Theorem 2.2.3])**.** Let $(\mathcal{C}, J)$ be a site with a Grothendieck coverage and $\mathcal{C}' \hookrightarrow \mathcal{C}$ a J-dense full subcategory. Then the restriction functor $\mathrm{res} : \mathsf{Pre}(\mathcal{C}) \to \mathsf{Pre}(\mathcal{C}')$ itself restricts to a functor $\mathrm{res} : \mathsf{Sh}(\mathcal{C}, J) \to \mathsf{Sh}(\mathcal{C}', J|_{\mathcal{C}'})$, and this functor is an equivalence of categories.

Note that $\tau(j^{\mathsf{Cart}}_{\text{open}}) = \tau(j^{\mathsf{Man}}_{\text{open}})|_{\mathsf{Cart}}$. This can be seen by simply noting that every sieve in $\tau(j^{\mathsf{Man}}_{\text{open}})|_{\mathsf{Cart}}$ is generated by a open cover by cartesian spaces, and contains every such sieve. A similar argument proves the same for $j^{\mathsf{Cart}}_{\text{good}}$ and $j^{\mathsf{Open}}_{\text{good}}, j^{\mathsf{Open}}_{\text{open}}$.

Corollary 3.33. All categories of the form $\mathsf{ConSh}(\mathcal{C}, j)$ for $\mathcal{C} \in \{\mathsf{Cart}, \mathsf{Open}, \mathsf{Man}\}$ and $j \in \{j_{\text{open}}, j_{\text{good}}\}$ are equivalent.

Proof. Theorem 3.32 implies that the categories $\mathsf{Sh}(\mathcal{C}, j)$ for $\mathcal{C} \in \{\mathsf{Cart}, \mathsf{Open}, \mathsf{Man}\}$ and $j \in \{j_{\text{open}}, j_{\text{good}}\}$ are all equivalent. Further, using the same argument as in the proof of [WW14, Lemma 2.9], the above equivalences restrict to equivalences of all the full subcategories $\mathsf{ConSh}(\mathcal{C}, j)$ of concrete sheaves. □

Thus by Theorem 3.15, we have that $\mathsf{Diff}' \simeq \mathsf{ConSh}(\mathsf{Open}, j_{\text{open}})$, and by Corollary 3.33, we have that $\mathsf{ConSh}(\mathsf{Open}, j_{\text{open}}) \simeq \mathsf{ConSh}(\mathsf{Cart}, j_{\text{good}}) \simeq \mathsf{Diff}$. Thus we have proved the main proposition of this section.

Proposition 3.34. The category of classical diffeological spaces Diff' is equivalent to the category of diffeological spaces Diff introduced in Definition 2.2.

4. Smooth Higher Stacks

4.1. **Model Structures on Simplicial Presheaves.**

For this section, we assume the reader is comfortable with simplicial homotopy theory as in [GJ12] and model categories as in [Hir09].

Definition 4.1. Let sSet denote the category of simplicial sets, and $\mathsf{sPre}(\mathsf{Cart})$ the category whose objects are functors $X : \mathsf{Cart}^{op} \to \mathsf{sSet}$ and whose morphisms are natural transformations. We call such functors **simplicial presheaves**.

There is a fully faithful embedding $\mathsf{Set} \hookrightarrow \mathsf{sSet}$, which we denote by $S \mapsto {}^cS$, where $({}^cS)_n = S$ for all $n \geq 0$, and all of the face and degeneracy maps are the identity on S. Similarly there is a fully faithful embedding $\mathsf{Pre}(\mathsf{Cart}) \hookrightarrow \mathsf{sPre}(\mathsf{Cart})$, which we also denote by $F \mapsto {}^cF$, such that $({}^cF)(U) = {}^cF(U)$ for all $U \in \mathsf{Cart}$. We call simplicial presheaves of this form **discrete simplicial presheaves**. This functor has a left adjoint $\pi_0 : \mathsf{sPre}(\mathsf{Cart}) \to \mathsf{Pre}(\mathsf{Cart})$, defined objectwise by

$$(\pi_0 X)(U) = \operatorname{coeq}\left(X(U)_0 \leftleftarrows X(U)_1 \right),$$

and a right adjoint $(-)_0 : \mathsf{sPre}(\mathsf{Cart}) \to \mathsf{Pre}(\mathsf{Cart})$ defined objectwise by $X_0(U) = X(U)_0$. Note that limits and colimits in $\mathsf{sPre}(\mathsf{Cart})$ are computed objectwise.

There is also a functor $(-)_c : \mathsf{sSet} \to \mathsf{sPre}(\mathsf{Cart})$ defined objectwise by $X_c(U) = X$ for every $U \in \mathsf{Cart}$. We call simplicial presheaves of this form **constant simplicial presheaves**. The category of simplicial presheaves on Cart is simplicially enriched. Let

X and Y be simplicial presheaves, then let $\underline{\mathsf{sPre(Cart)}}(X,Y)$ denote the simplicially-enriched Hom, defined degreewise by

$$\underline{\mathsf{sPre(Cart)}}(X,Y)_n = \mathsf{sPre(Cart)}(X \times \Delta^n_c, Y).$$

Compare this with the simplicial mapping space for simplicial sets, namely if X and Y are simplicial sets, then let $\underline{\mathsf{sSet}}(X,Y)$ denote the simplicial set defined degreewise by $\underline{\mathsf{sSet}}(X,Y)_n = \mathsf{sSet}(X \times \Delta^n, Y)$.

If K is a simplicial set and X is a simplicial presheaf, then let X^K denote the simplicial presheaf which is defined objectwise by $(X^K)(U) = \underline{\mathsf{sSet}}(K, X(U))$. Then $\mathsf{sPre(Cart)}$ is tensored and cotensored over sSet in the sense that for simplicial presheaves X and Y and simplicial set K, there is the following natural isomorphism

$$\underline{\mathsf{sPre(Cart)}}(X \times K_c, Y) \cong \underline{\mathsf{sPre(Cart)}}(X, Y^K).$$

The category $\mathsf{sPre(Cart)}$ admits many model structures. Here we will discuss two of them. Say a map $f : X \to Y$ of simplicial presheaves is an **objectwise weak equivalence** if $f : X(U) \to Y(U)$ is a weak equivalence of simplicial sets for every $U \in \mathsf{Cart}$. Similarly a **objectwise fibration** is a map $f : X \to Y$ of simplicial presheaves such that $f : X(U) \to Y(U)$ is a Kan fibration of simplicial sets for every $U \in \mathsf{Cart}$.

Theorem 4.2 ([BK72, Page 314])**.** There is a cofibrantly generated, simplicial model structure, which we call the **projective model structure** or **Bousfield-Kan model structure** on $\mathsf{sPre(Cart)}$, whose weak equivalences are the objectwise weak equivalences, and whose fibrations are the objectwise fibrations.

Remark 4.3. In fact, the projective model structure makes $\mathsf{sPre(Cart)}$ a combinatorial model category, see [Lur09, Section A.2.6].

Remark 4.4. There is a Quillen equivalent model structure on simplicial presheaves where the cofibrations and weak equivalences are objectwise, which is called the injective or Heller model structure. See [Bla01] for an overview of the different model structures on simplicial presheaves.

As is often the case with model structures, while the descriptions of weak equivalences and fibrations in the projective model structure are convenient, the cofibrations of the projective model structure are less simple to describe. However, the following result gives a sufficient condition for a simplicial presheaf to be cofibrant.

Theorem 4.5 ([Dug01, Corollary 9.4])**.** A simplicial presheaf X is cofibrant in the projective model structure on simplicial presheaves if

(1) X is degreewise a coproduct of representables, i.e. $X_n = \coprod_{i \in I} yU_i$ for every $n \geq 0$,
(2) X is split, in the sense that as a functor $X : \mathcal{C}^{op} \to \mathsf{sSet}$ it factors through the category sSet_{nd} whose objects are simplicial sets and whose morphisms are those maps of simplicial sets that map non-degenerate simplices to non-degenerate simplices.

We say that X is a **projective cofibrant** simplicial presheaf.

Corollary 4.6. If $U \in \mathsf{Cart}$, then ${}^c yU$ is a projective cofibrant simplicial presheaf on Cart.

Example 4.7. If M is a finite dimensional smooth manifold, and $\mathcal{U} = \{U_i\}_{i \in I}$ is a good open cover, then consider the simplicial presheaf $\check{C}(\mathcal{U})$ defined in degree n by

$$\check{C}(\mathcal{U})_n = \prod_{i_0, \dots, i_{n-1}} y\left(U_{i_0} \cap \cdots \cap U_{i_{n-1}}\right),$$

with face and degeneracy maps given by inclusions of open sets. Since $\mathcal{U}$ is a good open cover, $\check{C}(\mathcal{U})$ is a projective cofibrant simplicial presheaf. We call it the **Čech nerve** on $\mathcal{U}$. There is a canonical map

$$\check{C}(\mathcal{U}) \xrightarrow{\pi} {}^c M,$$

of simplicial presheaves on Cart. However this map is not an objectwise weak equivalence in general.

If $U \in$ Cart, and $\mathcal{U} = \{U_i\}_{i \in I}$ is a good open cover of U, then we can consider the canonical map

$$\check{C}(\mathcal{U}) \xrightarrow{\pi} {}^c yU.$$

Let $\check{C}$ denote the class of such maps as U varies over Cart and $\mathcal{U}$ varies over good open covers of U.

Definition 4.8. Let $\mathsf{s}\check{\mathsf{P}}\mathsf{re}(\mathsf{Cart})$ denote the left Bousfield localization of the projective model structure on sPre(Cart) at the class of maps $\check{C}$. We call this the **Čech model structure** on sPre(Cart).

The Čech model structure is described in greater detail in [DHI04, Appendix A]. Since $\mathsf{s}\check{\mathsf{P}}\mathsf{re}(\mathsf{Cart})$ is a left Bousfield localization of the projective model structure, it inherits the same cofibrations, and therefore cofibrant objects. We call its weak equivalences the Čech weak equivalences. Note that any objectwise weak equivalence of simplicial presheaves will be a Čech weak equivalence.

We call the fibrant objects of this model structure **∞-stacks** on Cart. They can be characterized as follows.

Proposition 4.9. A simplicial presheaf X on Cart is an ∞-stack on Cart if and only if it is projective fibrant (objectwise a Kan complex), and if for every $U \in$ Cart and good open cover $\mathcal{U}$ of U, the map

$$\underline{\mathsf{sPre}(\mathsf{Cart})}(yU, X) \to \underline{\mathsf{sPre}(\mathsf{Cart})}(\check{C}(\mathcal{U}), X), \tag{11}$$

is a weak equivalence of simplicial sets. We say that X satisfies **Čech descent**.

Proof. This follows from the definition of left Bousfield localization. □

By a simplicially-enriched version of the Yoneda Lemma, $\underline{\mathsf{sPre}(\mathsf{Cart})}(yU, X) \cong X(U)$. Thus we wish to better understand the right hand side of (11). To do this we will exploit the following result.

Lemma 4.10. Let X be a simplicial presheaf. Then

$$X \cong \int^{[n] \in \Delta^{op}} \Delta^n_c \times {}^c X_n,$$

where the colimit is taken in the category of simplicial presheaves, Δ^n_c is the constant simplicial presheaf on Δ^n and ${}^cX_n$ is the discrete simplicial presheaf on the presheaf X_n.

Proof. This follows from the corresponding fact for simplicial sets [GJ12, Lemma I.2.1]. □

Thus we have

$$\check{C}(\mathcal{U}) \cong \int^{[n]\in\Delta^{op}} \Delta^n_c \times \coprod_{i_0\dots i_n} yU_{i_0\dots i_n}.$$

This implies that

$$\begin{aligned}\underline{\mathsf{sPre(Cart)}}(\check{C}(\mathcal{U}),X) &\cong \int_{[n]} \underline{\mathsf{sPre(Cart)}}(\Delta^n_c \times \coprod_{i_0\dots i_n} yU_{i_0\dots i_n}, X)\\ &\cong \int_{[n]} \prod_{i_0\dots i_n} \underline{\mathsf{sPre(Cart)}}\big(yU_{i_0\dots i_n}, X^{\Delta^n}\big)\\ &\cong \int_{[n]} \prod_{i_0\dots i_n} \underline{\mathsf{sSet}}\big(\Delta^n, X(U_{i_0\dots i_n})\big)\\ &\cong \int_{[n]} \underline{\mathsf{sSet}}\left(\Delta^n, \prod_{i_0\dots i_n} X(U_{i_0\dots i_n})\right).\end{aligned} \tag{12}$$

This kind of limit is special enough to have its own name.

Definition 4.11. Let F be a cosimplicial simplicial set, namely a functor $F : \Delta \to \mathsf{sSet}$. Then let $\mathsf{Tot}(F)$ denote the simplicial set given by the end

$$\mathsf{Tot}(F) = \int_{[n]\in\Delta} \underline{\mathsf{sSet}}(\Delta^n, F^n),$$

where $\underline{\mathsf{sSet}}(X,Y)$ denotes the simplicial mapping space between two simplicial sets X and Y, namely $\underline{\mathsf{sSet}}(X,Y)_n = \mathsf{sSet}(X\times\Delta^n, Y)$. For a cosimplicial simplicial set F, $\mathsf{Tot}(F)$ is often called the **total object** or **totalization** of F.

A more convenient way of looking at $\mathsf{Tot}(F)$ is as the simplicial mapping space $\underline{\mathsf{csSet}}(\Delta, F)$, where Δ denotes the cosimplicial simplicial set $[m]\mapsto\Delta^m$. In other words, an n-simplex of the simplicial set $\mathsf{Tot}(F)$ is a map of cosimplicial simplicial sets $\Delta\times\Delta^n \to F$. The full data of such a map is a commutative diagram of the form

$$\begin{tikzcd} \Delta^n \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \Delta^1\times\Delta^n \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \Delta^2\times\Delta^n \arrow[d] & \dots \\ F^0 \arrow[r, shift left] \arrow[r, shift right] & F^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2 & \dots \end{tikzcd}$$

where each arrow is a map of simplicial sets, and we've hidden the codegeneracy maps for clarity in the diagram.

Thus X is an ∞-stack if and only if X is projective fibrant, and the canonical map

$$X(U) \to \mathrm{Tot}\Big(X(\check{C}(\mathcal{U}))\Big) \tag{13}$$

is a weak equivalence of simplicial sets, where $X(\check{C}(\mathcal{U}))$ is the cosimplicial simplicial set defined degreewise by $X(\check{C}(\mathcal{U}))_n = \prod_{i_0 \dots i_n} X(U_{i_0 \dots i_n})$. This concrete description has a pleasing abstract description as well.

Proposition 4.12. If X is a projective fibrant (objectwise Kan) simplicial presheaf, $U \in \mathsf{Cart}$ and $\mathcal{U}$ is a good cover of U, then

$$\mathrm{Tot}\Big(X(\check{C}(\mathcal{U}))\Big) \simeq \mathrm{holim}_{[n]\in\Delta} \prod_{i_0 \dots i_n} X(U_{i_0 \dots i_n}), \tag{14}$$

where the right hand side is the homotopy limit over Δ taken in the Quillen model structure on simplicial sets.

Proof. By [Hir09, Theorem 18.7.4], $\mathrm{Tot}\Big(X(\check{C}(\mathcal{U}))\Big) \to \mathrm{holim} \prod_{i_0 \dots i_n} X(U_{i_0 \dots i_n})$ is a weak equivalence if $X(\check{C}(\mathcal{U}))$ is a Reedy fibrant cosimplicial simplicial set. By [Gla+22, Lemma C.5], if X is projective fibrant, then $X(\check{C}(\mathcal{U}))$ is Reedy fibrant. □

Thus by Proposition 4.12, if X is a projective fibrant simplicial presheaf, then it is an ∞-stack if and only if the canonical map

$$X(U) \to \mathrm{holim}_\Delta \left(\prod_i X(U_i) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \prod_{i,j} X(U_{ij}) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \prod_{i,j,k} X(U_{ijk}) \quad \cdots \quad \right) \tag{15}$$

is a weak equivalence of simplicial sets.

If $X = {}^cF$ is a presheaf of sets, then $\underline{\mathsf{sPre}(\mathsf{Cart})}(yU, {}^cF) \cong F(U)$, and

$$\underline{\mathsf{sPre}(\mathsf{Cart})}(\check{C}(\mathcal{U}), {}^cF) \cong \mathsf{Pre}(\mathsf{Cart})(\pi_0 \check{C}(\mathcal{U}), F) \cong \mathrm{eq}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \right).$$

The right hand side is the usual equalizer one sees in the definition of a sheaf of sets. Note that if $f : {}^cX \to {}^cY$ is a map of discrete simplicial sets, then f is a weak equivalence if and only if it is an isomorphism of sets. Thus ${}^cF$ is an ∞-stack if and only if the canonical map

$$F(U) \to \mathrm{eq}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \right),$$

is an isomorphism of sets for every $U \in \mathsf{Cart}$ and good open cover $\mathcal{U}$. In other words, for discrete simplicial presheaves, being an ∞-stack is equivalent to being a sheaf.

Suppose that $\pi : G \to \mathsf{Cart}$ is a category fibered in groupoids, then it is well known that G is a stack, in the classical sense, if and only if the map

$$G(U) \to \mathrm{holim} \left(\prod_i G(U_i) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \prod_{i,j} G(U_{ij}) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \prod_{i,j,k} G(U_{ijk}) \right),$$

is an equivalence of groupoids, where the right hand side is a homotopy limit of groupoids as described in [Car11, Section I.1.7]. Now consider the nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$. By [Hol08, Theorem 1.2] we can restrict to the case when G is a strict presheaf of groupoids, namely a functor $G : \mathsf{Cart}^{op} \to \mathsf{Gpd}$. Then NG is a simplicial presheaf that will be projective fibrant, and it will be an ∞-stack if and only if G is a stack in the classical sense. Thus the notion of ∞-stack simultaneously generalizes the notion of sheaf and stack, and provides all of the power the homotopy theory of simplicial sets has to offer.

The main example of stack we will consider in this paper is the following.

Example 4.13. Suppose G is a group object in $\mathsf{Sh}(\mathsf{Cart})$, which we will call a sheaf of groups. Consider the presheaf of groupoids on Cart that sends a $U \in \mathsf{Cart}$ to the groupoid

$$\mathbf{B}G(U) := [G(U) \rightrightarrows *],$$

where both source and target maps are the unique map to the singleton set. Thus there is a single object in this groupoid, and for every section $s \in G(U)$, there is an isomorphism from the unique object to itself.

Now if G is in particular a Lie group, then we can consider it as a sheaf of groups on Cart. Then $\mathbf{B}G(U)$ will be the groupoid $[C^\infty(U,G) \rightrightarrows *]$. Now since all principal G-bundles are trivial on Cartesian spaces, it is easy to see that there is an objectwise equivalence of groupoids

$$[C^\infty(U,G) \rightrightarrows *] \simeq \mathrm{Prin}_G(U),$$

where $\mathrm{Prin}_G(U)$ denotes the groupoid of principal G-bundles on U. Indeed, every object of the right hand groupoid is isomorphic to the trivial bundle $U \times G \to U$, and the automorphisms of a trivial bundle are in bijection with maps $U \to G$. In [Car11, Section I.2] it is proven that Prin_G is a stack (in the classical sense) on Man and Cart. The argument above proves that $\mathbf{B}G$ is also a stack (in the classical sense) on Cart. However $\mathbf{B}G$ is not a stack on Man. If we take the nerves of these presheaves of groupoids[2], $N\mathbf{B}G$ and $N\mathrm{Prin}_G$, then since they are nerves of classical stacks on Cart, they will be ∞-stacks on Cart, and they are objectwise weak equivalent as simplicial presheaves. See [FSS+12, Section 3.2] for more details.

We often drop the nerve N from the notation of our ∞-stacks, and we call $\mathbf{B}G$ the **delooping stack** of G.

One of the most useful aspects of simplicial model categories is being able to define homotopically invariant mapping spaces.

Definition 4.14. If X and A are simplicial presheaves, then define

$$\mathbb{R}\mathrm{Hom}(X,A) := \underline{\mathsf{sPre}(\mathsf{Cart})}(QX, RA) \tag{16}$$

where QX is a cofibrant replacement for X and RA is a fibrant replacement of A in the Čech model structure. We call $\mathbb{R}\mathrm{Hom}(X,A)$ the **derived mapping space** of X and A.

A key property of derived mapping spaces is their invariance under weak equivalence. Indeed suppose there is a Čech weak equivalence $f : X \to X'$, then the canonical map

$$\mathbb{R}\mathrm{Hom}(f,A) : \mathbb{R}\mathrm{Hom}(X',A) \to \mathbb{R}\mathrm{Hom}(X,A)$$

[2] Technically Prin_G is not a strict presheaf of groupoids, however we can replace it with an equivalent strict presheaf of groupoids.

is a weak equivalence of simplicial sets, similarly for a Čech weak equivalence $g : A \to A'$, see [Hir09, Chapter 17].

Many classical invariants of smooth manifolds can be cast in the language of derived mapping spacces.

Example 4.15. If M is a finite dimensional smooth manifold, and A is an abelian group, then let A_c denote the sheaf on $\mathcal{O}(M)$ of locally constant functions to A. Then [Bro73] proves that

$$H^k(M,A) \cong \pi_0 \mathbb{R}\mathrm{Hom}(M, A_c[k]),$$

where $H^k(M,A)$ is the kth classical abelian sheaf cohomology of M with values in A_c, and $A_c[k]$ denotes the kth delooping of A_c, which will define in Section 4.3.

Example 4.16. If G is a Lie group, and M is a finite dimensional smooth manifold, then [FSS+12, Page 23] proves that there is a weak equivalence of simplicial sets

$$N\mathrm{Prin}_G(M) \simeq \mathbb{R}\mathrm{Hom}(M, \mathbf{B}G),$$

where $N\mathrm{Prin}_G(M)$ denotes nerve of the groupoid of principal G-bundles on M.

Now if G is a diffeological group, and X is a diffeological space, we can consider them both as simplicial presheaves on Cart, and take $\mathbb{R}\mathrm{Hom}(X, \mathbf{B}G)$. It would be hoped that this would in some way reproduce diffeological principal G-bundles, in analogy to Example 4.16. One main goal of this paper is to prove that this is indeed the case. But first we must investigate $\mathbb{R}\mathrm{Hom}(X, \mathbf{B}G)$. If X was cofibrant, and $\mathbf{B}G$ were fibrant in the Čech model structure, then $\mathbb{R}\mathrm{Hom}(X, \mathbf{B}G)$ would be computable. However diffeological spaces are not projective cofibrant in general (though cartesian spaces are). Thus we must find a projective cofibrant simplicial presheaf QX which is Čech weak equivalent to X. This will be the subject of Section 4.2.

However, it is indeed the case that $\mathbf{B}G$ is fibrant, thanks to the following incredible theorem.

Theorem 4.17 ([SS21, Lemma 3.3.29], [Pav22, Proposition 4.13])**.** Let G be a sheaf of groups on Cart. Then $\mathbf{B}G$ is an ∞-stack on Cart.

Thus if G is a diffeological group, then it will in particular be a sheaf of groups on Cart, and therefore $\mathbf{B}G$ will be an ∞-stack.

4.2. **Resolutions of Diffeological Spaces.** Here we discuss three ways of "resolving" a diffeological space into a diffeological category. One of which, QX comes forth immediately from the projective model structure on simplicial presheaves. The other two, which we denote $\check{C}(X)$ and $B//M$, appear in [KWW21] and [Igl20] respectively, and are interesting in their own right. We compare these three resolutions in two homotopical contexts, namely by taking their nerves we can compare them as simplicial presheaves in the Čech model structure on simplicial presheaves, and as diffeological categories we can compare them in the homotopy theory of categories internal to a site as described in [Rob12].

Let us start by describing the resolution QX for a diffeological space X. Since $\mathrm{s\check{P}re(Cart)}$ is a combinatorial model category, a cofibrant replacement functor Q exists. However, we are in the lucky situation that there is a cofibrant replacement functor Q with a relatively simple form.

Lemma 4.18 ([Dug01, Lemma 2.7])**.** Given a diffeological space X, thought of as a discrete simplicial presheaf on Cart, its cofibrant replacement QX is given by the simplicial presheaf

$$QX = \int^{[n]\in\Delta} \Delta^n_c \times \left(\coprod_{U_{p_n}\to\cdots\to U_{p_0}\to X} yU_{p_n} \right)$$

Let us examine this coend formula in more detail. In degree k, we have

$$(QX)_k = \coprod_{U_{p_k}\to\cdots\to U_{p_0}\to X} yU_{p_k}$$

where the coproduct is taken over the set $(N\mathsf{Plot}(X))_k$, namely the set of k composable morphisms in the category of plots over X. We will let $N_k := (N\mathsf{Plot}(X))_k$.

The face maps are given as follows:

$$d_i(x_{p_k}, f_{k-1}, f_{k-2}, \dots, f_0) = \begin{cases} (f_{k-1}(x_{p_k}), f_{k-2}, \dots, f_0) & i = 0 \\ (x_{p_k}, f_{k-1}, \dots, f_{k-i-1}f_{k-i}, \dots, f_0) & 0 < i < k \\ (x_{p_k}, f_{k-1}, \dots, f_1) & i = k. \end{cases} \tag{17}$$

Degeneracies insert identity maps.

For convenience, we will denote the coproduct over all plots as

$$B := (QX)_0 = \coprod_{U_p \to X} U_p.$$

Notice that there is a canonical map

$$B \xrightarrow{\pi} X$$

given by $\pi(p, x) = p(x)$.

This map induces a map $\pi : QX \to {}^cX$ of simplicial presheaves and [Dug01, Lemma 2.7] proves that this is an objectwise weak equivalence, and thus a Čech weak equivalence. By construction we also have the following isomorphism of presheaves on Cart.

Lemma 4.19. If X is a diffeological space, then the map $\pi : QX \to {}^cX$ induces an isomorphism of presheaves on Cart

$$\pi_0 QX \cong X, \tag{18}$$

where $\pi_0 : \mathsf{sPre}(\mathsf{Cart}) \to \mathsf{Pre}(\mathsf{Cart})$ is defined in Section 4.

Proof. This follows from the fact that every presheaf is a colimit of representables, see the discussion above [Dug01, Lemma 2.7]. ☐

In low degrees, this simplicial presheaf/simplicial diffeological space looks like:

$$B = \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \leftleftarrows\!\!\!\rightarrow \coprod_{f_0 : U_{p_1} \to U_{p_0}} U_{p_1} \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; \coprod_{U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_2} \qquad \cdots$$

where the maps $f_i : U_{p_{i+1}} \to U_{p_i}$ are understood to be morphisms in the plot category $\mathsf{Plot}(X)$.

In fact, QX is completely determined by QX_1 and QX_0 in the following sense. Let $N : \mathsf{DiffCat} \to \mathsf{sDiff}$ denote the nerve functor from diffeological categories (categories internal to the category of diffeological spaces) to the category of simplicial diffeological spaces, defined degreewise by

$$NC_k = C_1 \times_{t,C_0,s} C_1 \times_{t,C_0,s} \cdots \times_{t,C_0,s} C_1,$$

where the iterated pullback is taken k-times, where each $C_1 \times_{t,C_0,s} C_1$ denotes the pullback with respect to the target and source maps respectively.

For a diffeological space X, the first two spaces and maps between them in QX, namely $[QX_1 \rightrightarrows B]$ forms a diffeological category. The source, target and unit maps are defined by the simplicial structure, but we recall their definitions here. The source map $s : QX_1 \to B$ is defined by $s(x_{p_1}, f_0) = x_{p_1}$ and its target map $t : QX_1 \to B$ is defined by $t(x_{p_1}, f_0) = f_0(x_{p_1})$. The unit map $u : B \to QX_1$ is defined by $u(x_p) = (x_p, 1_{U_p})$. The composition map $c : QX_1 \times_B QX_1 \to QX_1$ is defined by $c([x_{p_2}, f_1], [x_{p_1}, f_0]) = (x_{p_2}, f_0 \circ f_1)$. With this structure it is not hard to see that $[QX_1 \rightrightarrows B]$ is a diffeological category. In fact QX is completely determined by this diffeological category in the following sense.

Proposition 4.20. If X is a diffeological space, then

$$QX \cong N[QX_1 \rightrightarrows QX_0], \tag{19}$$

where we are thinking of QX as a simplicial diffeological space.

Proof. Let $\varphi : QX_k \to QX_1 \times_B QX_1 \times_B \cdots \times_B QX_1$ be the map defined as follows. Suppose that $(x_{p_k}, f_{k-1}, \dots, f_0) \in QX_k$. By induction, define $x_{p_{k-1}} = f_{k-1}(x_{p_k})$ and $x_{p_{k-n}} = f_{k-n}(x_{p_{k-n+1}})$ for $1 < n \leq k$. Then set

$$\varphi(x_{p_k}, f_{k-1}, \dots, f_0) = ([x_{p_k}, f_{k-1}], [x_{p_{k-1}}, f_{k-2}], \dots, [x_{p_1}, f_0]).$$

This map is smooth, as it is built out of projection maps. Now define $\psi : QX_1 \times_B QX_1 \times_B \cdots \times_B QX_1 \to QX_k$ as follows. A point of $QX_1 \times_B QX_1 \times_B \cdots \times_B QX_1$ is a collection of pairs $\{[x_{p_n}, f_{n-1}]\}_{1 \leq n \leq k}$ such that $f_{n-1}(x_{p_n}) = x_{p_{n-1}}$. Thus set

$$\psi([x_{p_k}, f_{k-1}], [x_{p_{k-1}}, f_{k-2}], \dots, [x_{p_1}, f_0]) = (x_{p_k}, f_{k-1}, \dots, f_0).$$

It is not hard to see that this map is smooth, and that φ and ψ are two-sided inverses for each other. $\square$

Lemma 4.21. If X is a diffeological space, then we can consider the coequalizer in Diff of $[QX_1 \rightrightarrows QX_0]$ and this is isomorphic to X, namely

$$X \cong \operatorname{coeq}\left(\coprod_{U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_1} \rightrightarrows \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \right). \tag{20}$$

Proof. Consider the relation $\sim$ on B given by $x_{p_1} \sim f_0(x_{p_1})$ if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots. This relation is reflexive and transitive, but it is not symmetric. Consider its symmetric closure, which we also denote by $\sim$. Thus $x_{p_1} \sim x_{p_0}$ if there exists a zig-zag of maps of plots connecting x_{p_1} and x_{p_0}. Now consider the map $\phi : B/\!\sim \to X$ given by

$[x_{p_1}] \mapsto p_1(x_{p_1})$. We wish to show that this is well defined. If there is a zig-zag of the form

$$\begin{tikzcd} U_{p_1} \arrow[dr, "p_1"'] & V_q \arrow[l, "f_0"'] \arrow[r, "f_1"] \arrow[d, "q"] & U_{p_0} \arrow[dl, "p_0"] \\ & X & \end{tikzcd}$$

namely a zig-zag of morphisms $x_{p_1} \xleftarrow{(y_q, f_0)} y_q \xrightarrow{(y_q, f_1)} x_{p_0}$ in QX_1, then $p_1(x_{p_1}) = p_1(f_0(y_q)) = q(y_q) = p_0(f_1(y_q)) = p_0(x_{p_0})$. All possible zig-zags will have this property as well, therefore $\phi : B/{\sim} \to X$ is well defined and it is not hard to see that it is smooth. Now if we let $q : B \to B/{\sim}$ denote the quotient map, then we have the following commutative diagram

$$\begin{tikzcd} B \arrow[rr, "q"] \arrow[dr, "\pi"'] & & B/{\sim} \arrow[dl, "\phi"] \\ & X & \end{tikzcd}$$

where both π and q are subductions. In fact π even has the property that if $p : U \to X$ is a plot, then it lifts globally to a map $\mathrm{in}_p : U \to B$. So $q\mathrm{in}_p : U \to B/{\sim}$ is a plot such that $\phi q \mathrm{in}_p = \pi \mathrm{in}_p = p$. Thus ϕ is a subduction. Now suppose $\phi[x_p] = \phi[y_q]$. Then $p(x_p) = q(y_q)$. Therefore there is a zig-zag of plot maps

$$\begin{tikzcd} & * \arrow[dl, "x_p"'] \arrow[dr, "y_q"] \arrow[dd] & \\ U_p \arrow[dr, "p"'] & & V_q \arrow[dl, "q"] \\ & X & \end{tikzcd} \tag{21}$$

where $*$ denotes the cartesian space $\mathbb{R}^0$. Thus $[x_p] = [y_q]$, so ϕ is injective. By [Igl13, Article 1.49], this implies that $\phi : B/{\sim} \to X$ is a diffeomorphism. □

Remark 4.22. Note that the coequalizer given in (18) is taken in the category $\mathsf{Pre}(\mathsf{Cart})$, which has different colimits than $\mathsf{ConSh}(\mathsf{Cart})$. Therefore it does not immediately imply Lemma 4.21. However, by combining the two results we have proven that $\pi_0 QX$ is isomorphic to the coequalizer of $QX_1 \rightrightarrows QX_0$ in the category of diffeological spaces.

Now as discussed in Section 4.1, if X is a diffeological space and G is a diffeological group, then we can consider the simplicial set $\mathbb{R}\mathrm{Hom}(X, \mathbf{B}G)$. By Theorem 4.17, we know that $\mathbf{B}G$ is fibrant, thus

$$\mathbb{R}\mathrm{Hom}(X, \mathbf{B}G) = \underline{\mathsf{sPre}(\mathsf{Cart})}(QX, \mathbf{B}G) \cong \underline{\mathsf{sPre}(\mathsf{Cart})}(N[QX_1 \rightrightarrows B], N[C^\infty(-, G) \rightrightarrows *]).$$

In Section 5 we will show that this simplicial set is weak equivalent to the nerve of the groupoid of diffeological principal G-bundles on X.

The fact that QX is the nerve of a diffeological category is interesting, as it allows us to compare it with other diffeological categories using the homotopy theory developed in [Rob12] for categories internal to a site. The site in this instance is the category Diff of diffeological spaces with the coverage of subductions, see Example 3.7. This homotopy theory provides us with a notion of weak equivalence $f : X \to Y$ of diffeological categories, see [Rob12, Definition 4.14], which if both X and Y are diffeological groupoids, coincides with the notion of weak equivalence of diffeological groupoids considered in [Wat22] and [Sch20].

If we consider X as a diffeological category $[X = X]$ with all structure maps being the identity, then the canonical map $[QX_1 \rightrightarrows QX_0] \to [X = X]$ of diffeological categories is not a weak equivalence, as it is not fully faithful.

However, there is another diffeological groupoid we can consider. Given a diffeological space X, we can consider the canonical map $\pi : B \to X$ as mentioned above. This can be made into a diffeological groupoid $\check{C}(X)$ by setting $\check{C}(X)_0 = B$ and $\check{C}(X)_1 = B \times_X B$, with the source and target maps being the obvious projection maps. We will call this the **Čech resolution** of X, as a diffeological groupoid. It is not hard then to check that the canonical map $\check{C}(X) \to [X = X]$ of diffeological groupoids is indeed a weak equivalence.

If we then take the nerve of $\check{C}(X)$, we obtain a simplicial diffeological space, which we can also consider as a simplicial presheaf.

Proposition 4.23. The natural map $\check{C}(X) \to {}^cX$ of simplicial presheaves is a Čech weak equivalence.

Proof. It is easily checked that the map $\pi : B \to X$ is a local epimorphism, as it is objectwise a surjection. Thus [DHI04, Corollary A.3] proves that $\check{C}(X) \to {}^cX$ is a weak equivalence in the Čech model structure on simplicial presheaves. □

Therefore we have a zig-zag of Čech weak equivalences of simplicial presheaves

$$\check{C}(X) \to {}^cX \leftarrow QX.$$

However $\check{C}(X)$ will not be cofibrant in the projective model structure on simplicial presheaves in general. Thus for our purposes, QX is the preferable resolution of X, while for the purposes of those interested in diffeological groupoid theory, $\check{C}(X)$ might be the more preferable resolution.

The final resolution we will discuss is that of the **gauge monoid** that appears in [Igl20]. Given a diffeological space X, its nebula $B = \coprod_{p \in \mathsf{Plot}(X)} U_p$ is a diffeological space, and we can consider the set of smooth maps $f : B \to B$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\pi"'] & & B \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

It inherits the subspace diffeology from the functional diffeology on $C^\infty(B,B)$. Notice that M acts on B by $B \times M \xrightarrow{\rho} B$, where $\rho(b,m) = m(b)$.

We can therefore consider the diffeological category $B//M := [B \times M \rightrightarrows B]$, where the source map $s : B \times M \to M$ is given by $s(b,m) = b$, and the target map $t : B \times M \to M$ is given by $t(b,m) = m(b)$.

There is a map $\delta : QX \to B//M$ defined as the identity on objects and on morphisms by $\delta(x_{p_1}, f_0) = (x_{p_1}, \delta f_0)$ where δf_0 denotes the map $\delta f_0 : B \to B$ that is the identity on every component U_p except for $p = p_1$, in which case $\delta f_0|_{U_{p_1}} = f_0$. It is not hard to check that this defines a map of diffeological categories.

In the reverse direction, there is a map $\mathrm{res} : B//M \to QX$ defined to be the identity on objects and on morphisms by

$$\mathrm{res}(x_p, m) = (x_p, m|_{U_p}).$$

It is not hard to see that the composition $QX \xrightarrow{\delta} B//M \xrightarrow{\mathrm{res}} QX$ is the identity, namely that QX is a retract of $B//M$.

There is a map $q : B//M \to \check{C}(X)$ described in [KWW21, Page 26] which is the identity on objects and on morphisms is defined by $q(x_p, m) = (x_p, m(x_p))$. This defines a map of diffeological categories.

To summarize, we have the following diagram of maps of diffeological categories, all of which are the identity on objects, but none of which are fully faithful.

$$\begin{tikzcd} QX \arrow[rr, shift left, "\delta"] \arrow[dr, "q\circ\delta"'] & & B//M \arrow[ll, shift left, "\mathrm{res}"] \arrow[dl, "q"] \\ & \check{C}(X) & \end{tikzcd} \tag{22}$$

The diffeological categories $\check{C}(X)$ and $B//M$ are used to construct Čech cohomology groups for diffeological spaces in [KWW21] and [Igl20].

4.3. **Diffeological Čech Cohomologies.** Here we will describe three notions of Čech cohomology for diffeological spaces that results from the material in Section 4.2.

Remark 4.24. In what follows we will always consider chain complexes and cochain complexes to be non-negatively graded, with differentials going down and up respectively.

Let A denote a diffeological abelian group. In [Igl20], Čech cohomology of a diffeological space X is defined[3] as follows. First consider $N(B//M)$, the simplicial diffeological space defined as the nerve of the diffeological category defined in section 4.2. Then

$$A^{N(B//M)_k} = A^{B\times M^{\times k}} = C^\infty(B \times M^{\times k}, A)$$

is precisely the diffeological space of smooth maps $B\times M^{\times k} \to A$. If we forget the smooth structure, then $C^\infty(N(B//M)_k, A)$ is an abelian group by pointwise addition. Thus we obtain a cosimplicial abelian group

$$A^B \rightrightarrows A^{B\times M} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} A^{B\times M\times M} \qquad \cdots$$

and from this one can obtain a cochain complex as follows.

[3] Modulo some details, Iglesias-Zemmour defines diffeological spaces with open subsets of cartesian spaces and uses a generating family of open balls, but they are clearly equivalent constructions. He also only restricts to discrete abelian diffeological groups.

If K is a cosimplicial abelian group, then we can define a cochain complex $C^{\mathrm{co}}K$ called the **associated cochain complex** by

$$(C^{\mathrm{co}}K)^n = K^n, \qquad d : (C^{\mathrm{co}}K)^n \to (C^{\mathrm{co}}K)^{n+1}, \qquad d = \sum_{i=0}^{n} (-1)^i d^i.$$

This definition extends to a functor $C^{\mathrm{co}} : \mathsf{cAb} \to \mathsf{CoCh}$, where cAb denotes the category of cosimplicial abelian groups and CoCh is the category of cochain complexes. Further there is a functorial direct sum decomposition as cochain complexes $C^{\mathrm{co}}K \cong N^{\mathrm{co}}K \oplus D^{\mathrm{co}}K$, where $D^{\mathrm{co}}K$ is the subcomplex consisting of degenerate simplices, and the inclusion $N^{\mathrm{co}}K \to C^{\mathrm{co}}K$ is a cochain homotopy equivalence of cochain complexes. We call $N^{\mathrm{co}}K \cong C^{\mathrm{co}}K/D^{\mathrm{co}}K$ the **normalized cochain complex** of K. This is a dual version of what is called the Dold-Kan correspondence, which is an adjoint equivalence

$$N : \mathsf{sAb} \rightleftarrows \mathsf{Ch} : \Gamma,$$

where N is the normalized chain complex functor and if V is a chain complex, then ΓV is defined degreewise by

$$\Gamma(V)_n = \bigoplus_{[n] \twoheadrightarrow [k]} V_k,$$

where the index is over all surjections $\varphi : [n] \to [k]$ in Δ. See [Wei95, Section 8.4] and [GJ12, Section III] for details.

The Iglesias-Zemmour Čech cohomology of X is then defined as the cohomology of this cochain complex:

$$\check{H}^k_{PIZ}(X, A) = H^k\left(N^{\mathrm{co}}\left[A^{B//M}\right]\right) \cong H^k\left(C^{\mathrm{co}}\left[A^{B//M}\right]\right).$$

Similarly, let $N\check{C}(X)$ denote the nerve of the Čech groupoid defined in Section 4.2. If A is an abelian diffeological group, then as above we can map $N\check{C}(X)$ into A to form a cosimplicial abelian group, and taking the cohomology of the associated cochain complex gives us the Krepski-Watts-Wolbert Čech cohomology [KWW21] of X with values in A:

$$\check{H}^k_{KWW}(X, A) = H^k\left(N^{\mathrm{co}}\left[A^{\check{C}(X)}\right]\right) \cong H^k\left(C^{\mathrm{co}}\left[A^{\check{C}(X)}\right]\right).$$

Theorem 4.25 ([KWW21, Theorem 5.15])**.** For $k = 1$, and an abelian diffeological group A, the Krepski-Watts-Wolbert Čech cohomology of a diffeological space X classifies diffeological principal A-bundles, in other words there is a bijection

$$\check{H}^1_{KWW}(X, A) \cong \pi_0 \mathsf{DiffPrin}_A(X),$$

where $\pi_0 \mathsf{DiffPrin}_A(X)$ denotes the set of isomorphism classes of diffeological principal A-bundles over X.

Remark 4.26. Theorem 4.25 still holds when A is nonabelian, see [KWW21, Remark 5.16].

We will now construct a cochain complex using the cofibrant replacement QX of a diffeological space X, for which an analogue of Theorem 4.25 holds.

Given an abelian group A and a non-negative integer n, there exists a simplicial set $K(A, n)$, called the n**th Eilenberg-Maclane space**, which has trivial homotopy groups in all degrees except for n, which has $\pi_n(K(A,n)) = A$. One can construct this simplicial set using the Dold-Kan correspondence. Namely consider the chain complex $A[k]$, defined by

$$(A[k])_n = \begin{cases} A & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}, \qquad d = 0.$$

Since $\Gamma(A[k])$ is a simplicial group, it will be a Kan complex, equipped with basepoint $*$ such that

$$\pi_n(\Gamma(A[k]), *) = \begin{cases} A & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Remark 4.27. For future reference, if V is a chain complex, then let $V[k]$ denote the chain complex such that $V[k]_n = V_{n-k}$, so that we identify an abelian group A with the chain complex $A[0]$, and then $(A[0])[k] = A[k]$.

Now we will define ∞-stack cohomology for simplicial presheaves. This theory, which we call ∞-stack cohomology, is very well developed, and generalizes many examples of cohomology found throughout mathematics, see [Sch13], [Lur09], [BNV16].

Definition 4.28. Let X be a projective cofibrant simplicial presheaf and A an ∞-stack. Then the zeroth ∞-**stack cohomology** of X with values in A is

$$H^0_\infty(X, A) := \pi_0 \mathbb{R}\mathrm{Hom}(X, A) \cong \pi_0 \underline{\mathsf{sPre(Cart)}}(QX, A).$$

Note that in the above definition, A is an arbitrary ∞-stack. Thus $H^0_\infty(X,A)$ is an example of nonabelian cohomology. However, in order to define H^1_∞, we must ask for more structure to A, namely that it be an ∞-stack, and that A also be a group object in sPre(Cart), namely that $A(U)$ be a simplicial group for each $U \in$ Cart and given a smooth map $f : U \to V$, the map $A(f) : A(V) \to A(U)$ is a map of simplicial groups. We call group objects of sPre(Cart) presheaves of simplicial groups.

Definition 4.29. Given a simplicial group G, let $\overline{W}G$ denote the simplicial set with

$$\begin{aligned} \overline{W}G_0 &= * \\ \overline{W}G_n &= G_{n-1} \times G_{n-2} \times \cdots \times G_0 \end{aligned} \tag{23}$$

with face and degeneracy maps given by

$$d_i(g_{n-1}, \dots, g_0) = \begin{cases} (g_{n-2}, \dots, g_0) & \text{if } i = 0 \\ (d_{i-1}(g_{n-1}), \dots, d_1(g_{n-i+1}), g_{n-i-1} \cdot d_0(g_{n-i}), g_{n-i-2}, \dots, g_0)) & \text{if } 1 \leq i \leq n \end{cases}$$

$$s_i(g_{n-1}, \dots, g_0) = \begin{cases} (1, g_{n-1}, \dots, g_0) & \text{if } i = 0 \\ (s_{i-1}(g_{n-1}), \dots, s_0(g_{n-i}), 1, g_{n-i-1}, \dots, g_0) & \text{if } 1 \leq i \leq n. \end{cases}$$

Simplicial sets of the form $\overline{W}G$ classify what are called principal twisted cartesian products or PTCPs in [May92]. The combinatorial structure of $\overline{W}G$ may look complicated, but it has other equivalent descriptions that are more motivated, see [GJ12, Chapter V] and [Ste12].

Lemma 4.30 ([GJ12, Corollary 6.8])**.** If G is a simplicial group, then $\overline{W}G$ will be a Kan complex.

If ${}^cG$ a discrete simplicial group, i.e. a group, then $\overline{W}^cG \cong N[G \rightrightarrows *]$, the nerve of G thought of as a groupoid with one object. Thus $|\overline{W}^cG|$, the geometric realization of the delooping, is weak homotopy equivalent to the classifying space BG.

Now if A is a presheaf of simplicial groups, then we can apply $\overline{W}$ objectwise, and we will obtain a functor $\overline{W} : \mathsf{sPre}(\mathsf{Cart}, \mathsf{sGrp}) \to \mathsf{sPre}(\mathsf{Cart})$, where $\mathsf{sPre}(\mathsf{Cart}, \mathsf{sGrp})$ denotes the full subcategory of presheaves of simplicial groups. Further, by Lemma 4.30, $\overline{W}A$ will be projective fibrant, i.e. objectwise a Kan complex.

Lemma 4.31. Let G be a sheaf of groups on Cart. Then the delooping stack of Example 4.13 is isomorphic to its delooping as a presheaf of simplicial groups

$$\mathbf{B}G \cong \overline{W}^cG.$$

So suppose that A is an ∞-stack on Cart, and further, that it is a presheaf of simplicial groups. Then we get a new simplicial presheaf $\overline{W}A$, and it will be projective fibrant. We therefore define the first ∞-stack cohomology group of a simplicial presheaf X with values in A to be

$$H^1_\infty(X, A) \cong \pi_0 \mathbb{R}\mathrm{Hom}(X, \overline{W}A).$$

In order to be able to compute this group, it would be convenient to know that $\overline{W}A$ is fibrant in the Čech model structure, i.e. is an ∞-stack. This follows thanks to the following incredible theorem.

Theorem 4.32 ([SS21, Proposition 3.3.30], [Pav22, Proposition 4.13])**.** If A is an ∞-stack on Cart that is also a presheaf of simplicial groups, then $\overline{W}A$ will be an ∞-stack on Cart.

Thus if A is an ∞-stack, then for any simplicial presheaf X, $H^0_\infty(X, A)$ is well defined, and if A is also a presheaf of simplicial groups, then $H^1_\infty(X, A)$ is also well defined. To obtain higher cohomology groups, we must ask for higher deloopings of A to exist.

Definition 4.33. Let A be an ∞-stack that is also a presheaf of simplicial groups. If $\overline{W}^kA$ is a presheaf of simplicial groups for all $1 \leq k \leq n-1$, and X is a simplicial presheaf, then define the nth ∞-stack cohomology of X with values in A as

$$H^n_\infty(X, A) = \pi_0 \mathbb{R}\mathrm{Hom}(X, \overline{W}^nA).$$

It is thus important to know under what conditions will these higher deloopings $\overline{W}^nA$ exist.

Lemma 4.34. If A is a simplicial abelian group, namely A is a simplicial group and A_k is an abelian group for all k, then $\overline{W}A$ will be a simplicial group, and further it will be an abelian simplicial group.

Proof. It follows from the isomorphism $\overline{W}A \cong TNA$ of [Ste12, Lemma 5.2] and the discussion of T in [AM66, Section III] that $\overline{W}A$ is a simplicial group, and that it is abelian is clear from the formula (23). □

Thus if A is a simplicial abelian group, $\overline{W}^kA$ exists for all k.

Lemma 4.35 ([Jar97, Section 4.6])**.** Let A be a simplicial abelian group. Then there is an isomorphism of chain complexes

$$N\overline{W}A \cong (NA)[1]$$

where $(NA)[1]$ is the chain complex NA shifted up by 1, i.e. $(NA[1])_k = (NA)_{k-1}$.

Lemma 4.36. If A is an abelian group, thought of as a discrete simplicial abelian group ${}^cA$, then $\overline{W}^k A$ exists for every $k \geq 0$, and there exists an isomorphism

$$\overline{W}^{k\,c}A \cong \Gamma(A[k])$$

Proof. We proceed by induction. For the base case, we have

$$N\overline{W}^c A \cong (N^c A)([1]).$$

But $N^c A \cong A[0]$ as is easily checked, and $(A[0])[1] = A[1]$, so $N\overline{W}^c A \cong A[1]$, thus

$$\Gamma N\overline{W}^c A \cong \overline{W}^c A \cong \Gamma A[1].$$

Now suppose $\overline{W}^{k-1\,c}A \cong \Gamma A[k-1]$. Then by Lemma 4.35

$$N\overline{W}\left(\overline{W}^{k-1\,c}A\right) \cong (N\Gamma A[k-1])[1]$$

but $N\Gamma A[k-1] \cong A[k-1]$ since N and Γ form an adjoint equivalence, thus:

$$N\overline{W}\left(\overline{W}^{k-1\,c}A\right) \cong N\overline{W}^{k\,c}A \cong A[k-1]([1]) = A[k]$$

taking the adjoint gives

$$\overline{W}^{k\,c}A \cong \Gamma A[k].$$

□

Now if we let A be an abelian diffeological group, then it will be an ∞-stack on Cart, since it is a sheaf on Cart, and it will be a presheaf of simplicial groups. Thus by 4.36 and 4.32, $\overline{W}^n A$ exists, and the nth ∞-stack cohomology of a diffeological space X with values in A is given by

$$H^n_\infty(X, A) = \pi_0 \underline{\mathsf{sPre(Cart)}}(QX, \overline{W}^n A) \cong \pi_0 \mathsf{Tot}(\overline{W}^n[A(QX)]), \tag{24}$$

where Tot is the totalization of Definition 4.11, $A(QX)$ is the cosimplicial abelian group which in degree k is given by $C^\infty(QX_k, A)$, and $\overline{W}^n[A(QX)]$ is $\overline{W}^n$ applied to $A(QX)$ degreewise. Now $\overline{W}^n[A(QX)] \cong \Gamma[A(QX)[n]]$ by Lemma 4.36.

Proposition 4.37 ([Jar16, Lemma 19])**.** If A is a cosimplicial abelian group, then

$$\pi_0 \mathsf{Tot}(\Gamma A[k]) \cong H^k(N^{\mathrm{co}} A) \tag{25}$$

where $\Gamma A[k]$ denotes the cosimplicial simplicial abelian group obtained by considering the abelian group A^i as a simplicial abelian group $\Gamma(A^i[k])$.

So substituting for the cosimplicial abelian group $A(QX)$ in Proposition 4.37 we have the main result of this section, which provides a concrete way of computing the ∞-stack cohomology of a diffeological space with values in an abelian diffeological group.

Corollary 4.38. If X is a diffeological space, and A is an abelian diffeological group, and we consider the cochain complex

$$C^{co}[A(QX)] = A^B \xrightarrow{d} A^{QX_1} \xrightarrow{d} A^{QX_2} \xrightarrow{d} \dots$$

then the nth ∞-stack cohomology of X with values in A can be computed by $C^{co}[A(QX)]$, namely

$$H^n_\infty(X,A) \cong H^n_\infty(N^{co}[A(QX)]) \cong H^n_\infty(C^{co}[A(QX)]).$$

This explicit description of ∞-sheaf cohomology will be useful in comparing the various Čech cohomologies.

Proposition 4.39. For a diffeological space X, and a diffeological abelian group A, the ∞-sheaf cohomology and Iglesias-Zemmour cohomology agree in degree 0:

$$H^0_{PIZ}(X,A) = H^0_\infty(X,A).$$

Proof. The set of 0-cocycles in ∞-sheaf cohomology is the set

$$H^0_\infty(X,A) = \left\{\tau : B \to A \,|\, \text{if } f : U_p \to U_q \text{ is a map of plots then } \tau \circ \delta f = \tau\right\},$$

where δf denotes the map $\delta f : B \to B$ that is the identity on every component except for U_p, where it is f. Equivalently it is the set of smooth maps $\tau : B \to A$ such that if $(x_{p_1}, f_0) \in QX_1$, then $\tau(f_0(x_{p_1})) = \tau(x_{p_1})$. The set of 0-cocycles in Iglesias-Zemmour cohomology is

$$H^0_{PIZ}(X,A) = \{\sigma : B \to A \,|\, \text{if } m \in M, \text{then } \sigma \circ m = \sigma\}.$$

Equivalently it is the set of smooth maps $\sigma : B \to A$ such that if $(x_{p_1}, m) \in B \times M$, then $\sigma(m(x_{p_1})) = \sigma(x_{p_1})$. Notice that $H^0_{PIZ}(X,A) \subseteq H^0_\infty(X,A)$, since every δf is an element of M. Now if $\tau \in H^0_\infty(X,A)$, and $(x_{p_1}, m) \in B \times M$, then $\tau(m(x_{p_1})) = \tau(x_{p_1})$, because $m|_{U_{p_1}}$ is a map of plots. Thus $H^0_\infty(X,A) \subseteq H^0_{PIZ}(X,A)$. $\square$

Note that

$$\pi_0 \underline{\mathsf{sPre(Cart)}}(QX, {}^cA) \cong \mathsf{Pre(Cart)}(\pi_0 QX, A) \cong \mathsf{Pre(Cart)}(X,A) \cong \mathsf{Diff}(X,A), \tag{26}$$

where the first isomorphism follows from the adjunction described in Section 4, and the second isomorphism follows from Remark 4.22.

Corollary 4.40. If X is a diffeological space, and A a diffeological abelian group, then

$$H^0_\infty(X,A) \cong H^0_{PIZ}(X,A) \cong H^0_{KWW}(X,A) \cong \mathsf{Diff}(X,A). \tag{27}$$

Proof. This follows from (26), Proposition 4.39 and [KWW21, Proposition 4.6]. $\square$

Now recall the map $q \circ \delta : QX \to \check{C}(X)$ from (22). This induces a map on cohomology which we will denote by $\varphi : H^\bullet_{KWW}(X,A) \to H^\bullet_\infty(X,A)$. In degree 1, φ has the following explicit description on cocycles. Namely if $\tau : \check{C}(X)_1 \to A$ is a cocycle, then $(\varphi\tau)(x_{p_1}, f_0) = \tau(x_{p_1}, f_0(x_{p_1}))$.

Proposition 4.41. For any diffeological space X and abelian diffeological group A, the map $\varphi : H^1_{KWW}(X,A) \to H^1_\infty(X,A)$ is an isomorphism.

Proof. Let us show that φ is surjective. Suppose that σ is a 1-cocycle for the cochain complex A^{QX}. This means that if f_1, f_0 are composable maps of plots, then

$$\sigma(f_1(x_{p_2}), f_0) = \sigma(x_{p_2}, f_0 f_1) - \sigma(x_{p_2}, f_1).$$

Now if $(x_{p_1}, f_0) \in QX_1$, then notice we have the following commutative diagram of plot maps

$$\begin{tikzcd}
& * \arrow[dl, "x_{p_1}"'] \arrow[dr, "f_0(x_{p_1})"] & \\
U_{p_1} \arrow[rr, "f_0"] \arrow[dr, "p_1"'] & & U_{p_0} \arrow[dl, "p_0"] \\
& X &
\end{tikzcd}$$

which implies that if σ is a 1-cocycle that $\sigma(x_{p_1}, f_0) = \sigma(*, f_0(x_{p_1})) - \sigma(*, x_{p_1})$. So consider the map $\tau : \check{C}(X)_1 \to A$ defined as follows. If $(x_p, y_q) \in \check{C}(X)_1$, then let $\tau(x_p, y_q) = \sigma(*, y_q) - \sigma(*, x_p)$. Then $(\varphi\tau)(x_{p_1}, f_0) = \sigma(x_{p_1}, f_0)$ for every $(x_{p_1}, f_0) \in QX_1$.

Now we wish to show that φ is injective. Suppose $\tau, \tau' : \check{C}(X)_1 \to A$ are 1-cocycles such that there exists some $\alpha : B \to A$ such that for every $(x_{p_1}, f_0) \in QX_1$,

$$\tau(x_{p_1}, f_0(x_{p_1})) - \tau'(x_{p_1}, f_0(x_{p_1})) = \alpha(f_0(x_{p_1})) - \alpha(x_{p_1}).$$

Then if $(x_p, y_q) \in \check{C}(X)_1$, we have the following commutative diagram of plot maps

$$\begin{tikzcd}
& * \arrow[dl, "x_p"'] \arrow[dr, "y_q"] \arrow[dd, "z"] & \\
U_p \arrow[dr, "p"'] & & V_q \arrow[dl, "q"] \\
& X &
\end{tikzcd}$$

where $z = p(x_p) = q(y_q)$, and we use z to refer to the point $*$ in the plot $z : * \to X$. Now since τ and τ' are 1-cocycles, it follows that

$$\tau(x_p, y_q) = \tau(z, y_q) - \tau(z, x_p), \qquad \tau'(x_p, y_q) = \tau(z, y_q) - \tau(z, x_p).$$

Therefore

$$\begin{aligned}
\tau(x_p, y_q) - \tau'(x_p, y_q) &= (\tau(z, y_q) - \tau(z, x_p)) - (\tau'(z, y_q) - \tau'(z, x_p)) \\
&= (\tau(z, y_q) - \tau'(z, y_q)) - (\tau(z, x_p) - \tau'(z, x_p)) \\
&= (\alpha(y_q) - \alpha(z)) - (\alpha(x_p) - \alpha(z)) \\
&= \alpha(y_q) - \alpha(x_p).
\end{aligned}$$

which means that τ and τ' differ by a coboundary in $A^{\check{C}(X)}$, so φ is injective. $\square$

5. Principal Diffeological Bundles as Principal Infinity Bundles

Principal Infinity Bundles were defined in [NSS14a] and [NSS14b]. Here, we compare this abstract notion to diffeological principal bundles.

Definition 5.1. A **diffeological group** is a group G equipped with a diffeology $\mathcal{D}_G$ such that the multiplication map $m : G\times G \to G$, and inverse map $i : G \to G$ are smooth.

Definition 5.2. A right **diffeological group action** of a diffeological group G on a diffeological space X is a smooth map $\rho : X\times G \to X$ such that $\rho(x, e_G) = x$, and $\rho(\rho(x, g), h) = \rho(x, gh)$, where e_G denotes the identity element of G.

Definition 5.3. Let G be a diffeological group, and P be a diffeological right G-space. A map $\pi : P \to X$ of diffeological spaces is a **diffeological principal G-bundle** if:

(1) the map $\pi : P \to X$ is a subduction, and
(2) the map $\mathrm{act} : P\times G \to P\times_X P$ defined by $(p, g) \mapsto (p, p\cdot g)$, which we call the **action map** is a diffeomorphism.

A map of diffeological principal G-bundles $P \to P'$ over X is a diagram

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

where f is a G-equivariant smooth map. A diffeological principal G-bundle is said to be **trivial** if it is isomorphic to the diffeological principal G-bundle $\mathrm{pr}_1 : U\times G \to U$. Let $\mathsf{DiffPrin}_G(X)$ denote the category of diffeological principal G-bundles over a diffeological space X.

The following are some properties of diffeological principal G-bundles that are not hard to prove, see [Igl13, Chapter 8].

Lemma 5.4. Let G be a diffeological group and $\pi : P \to X$ a diffeological principal G-bundle, then we have the following:

(1) if $f : Y \to X$ is a smooth map, then the pullback $f^*P \to Y$ is a diffeological principal G-bundle,
(2) if there is a section $s : X \to P$, namely $\pi\circ s = 1_X$, then P is trivial,
(3) if $f : P \to P'$ is a map of diffeological principal G-bundles over a diffeological space X, then it is an isomorphism.

By Lemma 5.4.(3), the category $\mathsf{DiffPrin}_G(X)$ is a groupoid for every diffeological group G and every diffeological space X.

Lemma 5.5. Given a diffeological group G and a diffeological right G-space P, a map $\pi : P \to X$ is a diffeological principal G-bundle if and only if for every plot $q : U \to X$, the pullback q^*P is trivial, and the map $\mathrm{act} : P\times G \to P\times_X P$ defined by $(p, g) \mapsto (p, p\cdot g)$ is a diffeomorphism.

Proof. ($\Rightarrow$) If $q : U \to X$ is a plot, then by Lemma 5.4 q^*P is a diffeological principal G-bundle over U. Since $U \cong \mathbb{R}^n$, by [Igl13, Article 8.19], q^*P is trivial.

($\Leftarrow$) If q^*P is trivial on every plot, then we have the following commutative diagram

$$\begin{tikzcd}
U \times G \arrow[r, "\phi"] \arrow[dr] & q^*P \arrow[r, "q'"] \arrow[d] & P \arrow[d, "\pi"] \\
 & U \arrow[r, "q"'] \arrow[ul, bend left, "1_U \times e_G"] & X
\end{tikzcd}$$

which means that $q' \circ \phi \circ 1_U \times e_G$ is a section of π. Since this is true for every plot $q : U \to X$, the map π is a subduction. $\square$

Lemma 5.6. Condition (2) of Definition 5.3 is equivalent to G acting on the fibers of π freely and transitively.

Proof. If $\mathrm{act} : P \times G \to P \times_X P$ is a diffeomorphism, then G clearly acts on the fibers of π freely and transitively. Now suppose G acts on the fibers of π freely and transitively. The map $\mathrm{act} : P \times G \to P \times_X P$ is smooth. Since the action is free and transitive it means also that the map act is a bijection. We need then only to show that the inverse function is smooth. Namely if $\langle q, q' \rangle : U \to P \times_X P$ is a plot, where $q, q' : U \to P$ are plots, then we wish to show that the composite function $\mathrm{act}^{-1}\langle q, q' \rangle$ is a plot of $P \times G$. Now this is the set map $u \mapsto (q(u), \mathrm{diff}(q(u), q'(u)))$, where $\mathrm{diff} : P \times_X P \to G$ is the composite map $\mathrm{proj}_2 \mathrm{act}^{-1}$. This is a plot of $P \times G$ if it is a plot in both factors. Obviously it is a plot of the first factor, so we need only show that it is a plot of the second factor. This is the map that we will denote by τ, namely $\tau(u) = \mathrm{diff}(q(u), q'(u))$. Now, let $r = \pi q = \pi q'$ denote the plot $r : U \to X$. We have the following commutative cube

$$\begin{tikzcd}
U \arrow[rrrr, bend left, "\langle 1_U, q' \rangle"] \arrow[dr, dashed, "h"] \arrow[dddr, "\langle q, 1_U \rangle"'] & & & & U \times G \arrow[dr, "\varphi_{q'}"] \arrow[dd] & \\
 & (U \times G) \times_U (U \times G) \arrow[urrr] \arrow[dd] \arrow[drr, "k"] & & & & P \arrow[dd, "\pi"] \\
 & & & P \times_X P \arrow[urr] \arrow[dd] & U \arrow[dr, "r"] & \\
 & U \times G \arrow[urrr] \arrow[drr, "\varphi_q"'] & & & & X \\
 & & & P \arrow[urr, "\pi"'] & &
\end{tikzcd} \tag{28}$$

and $(U \times G) \times_U (U \times G) \cong U \times G \times G$. Note that $\langle q, q' \rangle : U \to P \times_X P$ factors as kh, where k is the map $U \times G \times G \to P \times_X P$ induced by the plotwise trivializations of P along q and q'. Now we can see that τ factors as

$$\begin{tikzcd}
U \arrow[r, "\langle q, q' \rangle"'] \arrow[rrr, bend left, "\tau"] \arrow[dr, "h"'] & P \times_X P \arrow[r, "\mathrm{act}^{-1}"'] & P \times G \arrow[r, "\mathrm{proj}_2"'] & G \\
 & U \times G \times G \arrow[u, "k"] \arrow[r, "n"'] & U \times G \arrow[ur, bend right, "\mathrm{proj}_2"'] &
\end{tikzcd}$$

where $n : U \times G \times G \to U \times G$ is the smooth map $(u, g, g') \mapsto (u, (g')^{-1}g)$. Thus τ is a composite of smooth maps, and therefore is a plot. $\square$

Diffeological principal G-bundles are a true generalization of classical principal G-bundles in the following sense.

Proposition 5.7 ([Wal12, Theorem 3.1.7])**.** If M is a finite dimensional smooth manifold, G is a Lie group, and $\mathsf{Prin}_G(M)$ denotes the category whose objects are classical principal G-bundles over M and morphisms are G-equivariant bundle maps, then

$$\mathsf{Prin}_G(M) \simeq \mathsf{DiffPrin}_G(M).$$

Namely, the category of classical principal G-bundles over M and the category of diffeological principal G-bundles over M are equivalent.

Classically, principal G-bundles over a smooth manifold M are classified up to isomorphism by homotopy classes of maps from M to a classifying space BG. There has been recent work [CW21], [MW17] extending this result to diffeology. Since diffeological spaces are so much more general than smooth manifolds, one needs to only consider numerable principal G-bundles to classify them in this sense as in the above references.

However, there is another way of classifying principal G-bundles over a smooth manifold M, that produces the whole groupoid of principal G-bundles, rather than just the isomorphism classes, see Example 4.13. One goal of this paper is to extend this result to diffeological principal G-bundles.

Now, we turn to the notion of G-principal ∞-bundles. First, we need the following definitions, which will only be used in Definition 5.10.

Definition 5.8 ([DHI04, Section 3])**.** A map $f : X \to Y$ of simplicial presheaves on Cart is a **local fibration** if for every $U \in \mathsf{Cart}$, there exists a good open cover $\{U_i \subseteq U\}$ such that for every element U_i of the good open cover, there is a lift in every commutative diagram of the following form.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X(U) \arrow[r] & X(U_i) \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[urr, dashed] & Y(U) \arrow[r] & Y(U_i) \end{tikzcd} \tag{29}$$

Note that an objectwise fibration of simplicial presheaves is a local fibration. We say that a simplicial presheaf X is **locally fibrant** if the unique map $X \to *$ is a local fibration.

Definition 5.9 ([DI04, Theorem 6.15])**.** A map $f : X \to Y$ of simplicial presheaves on Cart is a **local weak equivalence** if for every $U \in \mathsf{Cart}$, there exists a good open cover $\{U_i \subseteq U\}$ such that for every element U_i of the good open cover, there is a dotted arrow in every commutative diagram of the following form,

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & RX(U) \arrow[r] & RX(U_i) \arrow[d, "Rf"] \\ \Delta^n \arrow[r] \arrow[urr, dashed] & RY(U) \arrow[r] & RY(U_i) \end{tikzcd} \tag{30}$$

where R is a fibrant replacement functor for sSet and the top left triangle commutes strictly, while the bottom right triangle commutes up to a homotopy relative to $\partial\Delta^n \hookrightarrow \Delta^n$.

Note that an objectwise weak equivalence is a local weak equivalence, and [DHI04] proves that Čech weak equivalences are local weak equivalences.

Definition 5.10 ([NSS14b, Definition 3.79])**.** Let G be a presheaf of simplicial groups acting on a simplicial presheaf P by $\rho : P \times G \to P$. Then a map $\pi : P \to X$ is a G-**Principal ∞-bundle**[4] if:

(1) π is a local fibration,
(2) The action of G on P is fiberwise, namely $\rho(g,-)$ sends fibers to fibers, and
(3) the map
$$P \times G \to P \times_X P$$
given by
$$(p,g) \mapsto (p, \rho(p,g))$$
is a local weak equivalence.

A map $P \xrightarrow{f} P'$ of G-principal ∞-bundles over X is a map that is G-equivariant and commutes with the bundle projections. Namely, it is a map fitting into the following commutative diagram:

$$\begin{tikzcd}
P \times G \arrow[rr, "f \times 1_G"] \arrow[d, "\rho"'] & & P' \times G \arrow[d, "\rho'"] \\
P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\
& X &
\end{tikzcd}$$

Let $\mathsf{Prin}^{\infty}_G(X)$ denote the category of G-principal ∞-bundles on X.

It is clear that if $\pi : P \to X$ is a diffeological principal G-bundle, then it is a G-principal ∞-bundle, when we think of X, G and P as discrete simplicial presheaves. This is because all maps between discrete simplicial presheaves are local fibrations, and all diffeomorphisms between diffeological spaces are local weak equivalences. Note that $\mathsf{DiffPrin}_G(X)$ is a groupoid, while in general $\mathsf{Prin}^{\infty}_G(X)$ is not a groupoid. So while these categories are not equivalent, we will prove that their nerves are weak homotopy equivalent.

Proposition 5.11. Let X be a locally fibrant simplicial presheaf and G a presheaf of simplicial groups. Then, there is a weak homotopy equivalence of simplicial sets

$$\mathbb{R}\mathrm{Hom}(X, \mathbf{B}G) \simeq N\mathsf{Prin}^{\infty}_G(X). \tag{31}$$

[4] In [NSS14b], what we call principal ∞-bundles are known as weakly principal G-bundles. Also, they only define this for simplicial sheaves, but there are no problems extending the definition and all of the theorems in that paper to simplicial presheaves.

Proof. First if X and Y are locally fibrant simplicial presheaves, then combining [Low15, Lemma 6.4] with [Low15, Theorem 3.12] and [DK80, Corollary 4.7] proves that

$$\mathbb{R}\mathrm{Hom}(X,Y) \simeq N\mathrm{Cocycle}(X,Y)$$

where $\mathrm{Cocycle}(X,Y)$ is the cocycle category as defined in [Low15, Definition 3.1]. Then [NSS14b, Theorem 3.95] proves that

$$N\mathrm{Cocycle}(X,\mathbf{B}G) \simeq N\mathsf{Prin}^{\infty}_G(X).$$

Since all presheaves of simplicial groups are locally fibrant, combining these gives the desired result. □

Since all diffeological spaces are locally fibrant, if X is a diffeological space and G a diffeological group, to prove that $N\mathsf{DiffPrin}_G(X)$ is weak equivalent to $N\mathsf{Prin}^{\infty}_G(X)$, it suffices to show that $N\mathsf{DiffPrin}_G(X)$ is weak homotopy equivalent to $\mathbb{R}\mathrm{Hom}(X,\mathbf{B}G)$. Let us examine $\mathbb{R}\mathrm{Hom}(X,\mathbf{B}G)$ more deeply. In Section 4.2 we saw that this is equal to the simplicial set $\underline{\mathsf{sPre(Cart)}}(QX,\mathbf{B}G)$. Now, if we consider the definition of QX given in Lemma 4.18, then by the same computation as (12), we have

$$\underline{\mathsf{sPre(Cart)}}(QX,\mathbf{B}G) \cong \mathsf{Tot}(\mathbf{B}G(QX)). \tag{32}$$

A k-simplex of $\mathsf{Tot}(\mathbf{B}G(QX))$ contains a huge amount of information, but in this case, since $\mathbf{B}G$ is objectwise the nerve of a groupoid, most of this information will be redundant. Let us describe what a vertex of this simplicial set is. It is a map of cosimplicial simplicial sets $\Delta^\bullet \times \Delta^0 \cong \Delta^\bullet \to \mathbf{B}G(QX)$. This means it is a commutative diagram of the form[5]:

$$\begin{tikzcd}
\Delta^0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g^0"'] & \Delta^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "g^1"'] & \Delta^2 \arrow[d, "g^2"'] & \cdots \\
\prod_{\mathsf{Plot}(X)} \mathbf{B}G(U_{p_0}) \arrow[r, shift left] \arrow[r, shift right] & \prod_{N_1} \mathbf{B}G(U_{p_1}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \prod_{N_2} \mathbf{B}G(U_{p_2}) & \cdots
\end{tikzcd}$$

Let us unravel what this means. Firstly, g^0 contributes no information, as $\mathbf{B}G(U_{p_0})_0 = C^\infty(U_{p_0},*)$. However, g^1 is the data of maps $g^1(f_0) := g_{f_0} : U_{p_1} \to G$ for every map of plots $f_0 : U_{p_1} \to U_{p_0}$. Now g^2 is the data of a map $g^2(f_1,f_0) := g_{f_1,f_0} : U_{p_2} \to G \times G$ for every pair of composable maps of plots $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$. Let $g^2(f_1,f_0) = (h,k)$. The data of the above cosimplicial map insists that

$$(d^0 g^1)(f_1,f_0) = g^1(f_0)\circ f_1 = (g^2 d^0)(f_1,f_0) = d_0(g^2(f_1,f_0)) = d_0(h,k) = k$$
$$(d^1 g^1)(f_1,f_0) = g^1(f_0 f_1) = (g^2 d^1)(f_1,f_0) = d_1(g^2(f_1,f_0)) = d_1(h,k) = kh$$
$$(d^2 g^1)(f_1,f_0) = g^1(f_1) = (g^2 d^2)(f_1,f_0) = d_2(g^2(f_1,f_0)) = d_2(h,k) = h.$$

In other words

$$g_{f_0 f_1} = (g_{f_0}\circ f_1)\cdot g_{f_1}. \tag{33}$$

[5]Where we exclude the codegeneracy maps from the notation for clarity.

We can visualize this as a triangle:

$$\begin{tikzcd}
 & \bullet \arrow[dr, "g_{f_0}\circ f_1"] & \\
\bullet \arrow[ur, "g_{f_1}"] \arrow[rr, "g_{f_0 f_1}"'] \arrow[rr, phantom, bend left=20, "g^2(f_1,f_0)"] & & \bullet
\end{tikzcd}$$

which is filled in if the condition (33) holds. The codegeneracy maps will guarantee that if f is an identity map of plots $f = 1_{U_p}$, then $g_f = e_G$ is the constant map at the unit of G.

Now here's an important point: g^3 will provide no further data. We will explain why using the notion of coskeleton.

Definition 5.12. A simplicial set X is k**-coskeletal** if for every boundary $\partial\Delta^n \to X$, there exists a unique n-simplex $\Delta^n \to X$ making the following diagram commute:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur] &
\end{tikzcd}$$

for all $n > k$.

For any k, let $\mathsf{sSet}_{\leq k}$ denote the category of k-truncated simplicial sets, namely presheaves on the full subcategory $\Delta_{\leq k}$ of Δ whose objects are partial orders $[n]$ for $n \leq k$. There is a functor $\tau_k : \mathsf{sSet} \to \mathsf{sSet}_{\leq k}$ just given by forgetting the higher simplices of the simplicial set. This functor has a fully faithful left adjoint sk_k and a fully faithful right adjoint cosk_k. A simplicial set X is k-coskeletal if the unit of the adjunction $X \to \mathrm{cosk}_k(X)$ is an isomorphism. For more details see [GJ12, Section VII.1].

If $X = N(\mathcal{C})$ is the nerve of a category $\mathcal{C}$, then X will be 2-coskeletal [GJ12, Lemma I.3.5]. In our case $\mathbf{B}G(QX)$ is a cosimplicial simplicial set such that $\mathbf{B}G(QX_n)$ is the nerve of a groupoid and therefore 2-coskeletal for every n. Now as we've seen, the 3-simplex $g^3 \in \mathbf{B}G(QX_3)$ is required to satisfy that $\partial g^3 = (d^0 g^1, d^1 g^1, d^2 g^1, d^3 g^1)$. But that means we've just specified a 3-boundary in a 2-coskeletal simplicial set. Thus there exists a unique filler g^3. This of course continues, so that a vertex $g \in \mathsf{Tot}(\mathbf{B}G(QX))_0$ determines and is completely determined by g^1 and g^2.

Let us repeat the above analysis for a 1-simplex in $\mathsf{Tot}(\mathbf{B}G(QX))$. This is the data of a commutative diagram:

$$\begin{tikzcd}
\Delta^0\times\Delta^1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "h^0"'] & \Delta^1\times\Delta^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "h^1"'] & \Delta^2\times\Delta^1 \arrow[d, "h^2"'] & \cdots \\
\prod_{N_0}\mathbf{B}G(U_{p_0}) \arrow[r, shift left] \arrow[r, shift right] & \prod_{N_1}\mathbf{B}G(U_{p_1}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \prod_{N_2}\mathbf{B}G(U_{p_2}) & \cdots
\end{tikzcd}$$

Now unravelling this diagram, skipping some similar details, such a 1-simplex consists of the following data. If g and g' are 0-simplices in $\mathsf{Tot}(\mathbf{B}G(QX))$ consisting of collections of maps $\{g_f\}$ and $\{g'_f\}$, then a 1-simplex is a collection of maps $\{h_{p_0} : U_{p_0} \to G\}$ indexed by plots $p_0 : U_{p_0} \to X$ such that if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots, then

$$g'_{f_0} \cdot h_{p_1} = (h_{p_0} \circ f_0) \cdot g_{f_0}. \tag{34}$$

By the same reasoning as before, the rest of the diagram provides no further conditions on this data, as the maps $\Delta^k \times \Delta^1 \to \mathbf{B}G(QX_k)$ will consist of $(k+1)$-simplices, and $\mathbf{B}G(QX_k)$ is 2-coskeletal, so that h depends only on h^0 and h^1. Namely given h^0 and h^1, the h^k for $k > 1$ are fully determined.

A 2-simplex in $\mathsf{Tot}(\mathbf{B}G(QX))$ will similarly be completely determined by its boundary. Similar reasoning also proves that there are no additional conditions coming from higher k-simplices of $\mathsf{Tot}(\mathbf{B}G(QX))$. In other words, $\mathsf{Tot}(\mathbf{B}G(QX))$ is 2-coskeletal. Further, since $\mathsf{s\check{P}re(Cart)}$ is a simplicial model category and $\mathbb{R}\mathsf{Hom}(X, \mathbf{B}G) \cong \mathsf{Tot}(\mathbf{B}G(QX))$, this implies that $\mathsf{Tot}(\mathbf{B}G(QX))$ is a Kan complex. This combined with the fact that it is 2-coskeletal implies that for any basepoint g, $\pi_k(\mathsf{Tot}(\mathbf{B}G(QX)), g) = 0$ for $k > 1$.

Now that we have an explicit description of $\mathsf{Tot}(\mathbf{B}G(QX))$ it is clear that this is nothing more than a diffeological version of the cocycle construction from classical differential geometry. Let us formalize this now.

Definition 5.13. If X is a diffeological space and G is a diffeological group, then call a collection $\{g_{f_0}\}$ of smooth maps $g_{f_0} : U_{p_1} \to G$ indexed by maps of plots of X satisfying (33) a **cocycle** of X with values in G. Given two cocycles, $g, g' : QX \to \mathbf{B}G$, we say a collection $\{h_{p_0}\}$ of smooth maps $h_{p_0} : U_{p_0} \to G$ indexed by plots of X satisfying (34) will be called a **morphism of cocycles**.

Let $\mathsf{Coc}(X, G)$ denote the category whose objects are cocycles of X with values in G and whose morphisms are morphisms of cocycles. Composition is defined as follows. If $h : g \to g'$ and $h' : g' \to g''$ are morphisms of cocycles, then let $(h' \circ h)$ denote the morphism of cocycles which is given by $(h' \circ h)_p = h'_p \cdot h_p$. Let us show that $(h' \circ h)$ is actually a morphism of cocycles. A morphism of cocycles $h : g \to g'$ implies $g'_{f_0} h_{p_1} = (h_{p_0} \circ f_0) g_{f_0}$ and $h' : g' \to g''$, implies $g''_{f_0} h'_{p_1} = (h'_{p_1} \circ f_0) g'_{f_0}$. Thus

$$g''_{f_0} h'_{p_1} = (h'_{p_0} \circ f_0)(h_{p_0} \circ f_0) g_{f_0} h^{-1}_{p_1}.$$

So $g''_{f_0}(h'_{p_1} \cdot h_{p_1}) = (h'_{p_0} \cdot h_{p_0} \circ f_0) g_{f_0}$. Thus $(h' \circ h)$ is a morphism of cocycles. Note that $\mathsf{Coc}(X, G)$ is a groupoid by taking $(h^{-1})_p = h_p^{-1}$, and therefore the nerve of this category is a Kan complex.

We want to construct a map $\Phi : \mathsf{Tot}(\mathbf{B}G(QX)) \to N\mathsf{Coc}(X, G)$. Consider the left adjoint $h : \mathsf{sSet} \to \mathsf{Cat}$ to the nerve functor N, that sends a simplicial set to its homotopy category [Rie14, Example 1.5.5], namely if X is a simplicial set, then hX is the category whose objects are the vertices of X, morphisms are freely generated by the 1-simplices of X and then quotiented by the 2-simplices, in the sense that if σ is a 2-simplex in X with $d_0\tau = x, d_1\tau = y, d_2\tau = z$, then $x \circ z = y$ in hX. Note that by unravelling the above definitions, the composition of two morphisms $h' \circ h$ in $h\mathsf{Tot}(\mathbf{B}G(QX))$ is given by multiplication $h' \cdot h$.

Let $\Phi : h\mathsf{Tot}(\mathbf{B}G(QX)) \to \mathsf{Coc}(X, G)$ denote the functor that sends an object $g = (g^0, g^1, \dots)$ to the cocycle it defines $\{g_{f_0}\}$, and a morphism $h = (h^0, h^1, \dots)$ to the morphism of cocycles it defines $\{h_p\}$. By the above discussion it is evident that this functor defines (one half of) an isomorphism of categories. From this the following result follows.

Lemma 5.14. There is an isomorphism of simplicial sets

$$N\mathsf{Coc}(X,G) \cong \mathsf{Tot}(\mathbf{B}G(QX)). \tag{35}$$

Now that we have an explicit description of the simplicial set $\mathsf{Tot}(\mathbf{B}G(QX))$, let us compare it with the groupoid $\mathsf{DiffPrin}_G(X)$ of diffeological principal G-bundles on X. We wish to show that there is an equivalence of categories $\mathsf{DiffPrin}_G(X) \simeq \mathsf{Coc}(X,G)$. Suppose that $g : QX \to \mathbf{B}G$ is a map of simplicial presheaves, which is equivalently a cocycle. We wish to obtain a diffeological principal G-bundle from this data.

Consider the diffeological groupoid

$$G \times G \overset{\mathrm{pr}_1}{\underset{m}{\rightrightarrows}} G$$

where the source map is the first projection and the target map is the multiplication map. This defines a strict presheaf of groupoids on Cart by

$$[U \mapsto (C^\infty(U, G\times G) \rightrightarrows C^\infty(U,G))].$$

Taking the nerve of this gives us a simplicial presheaf which we denote by $\mathbf{E}G$. There is a canonical map of simplicial presheaves $\mathbf{E}G \to \mathbf{B}G$ induced by the corresponding map of diffeological groupoids:

$$\begin{tikzcd} G \times G \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "m"'] \arrow[d, "s"'] & G \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & * \end{tikzcd}$$

where s is the map $(g_1, g_2) \mapsto g_2 g_1^{-1}$.

Remark 5.15. The simplicial presheaf $\mathbf{E}G$ and the map $\mathbf{E}G \to \mathbf{W}G$ described above are well known in the literature in the form $WG \to \overline{W}G$, which can be defined when G is any presheaf of simplicial groups, see [Ste12] or [NSS14b].

Now with such a map $g : QX \to \mathbf{B}G$, we can consider the following pullback in the category of simplicial presheaves.

$$\begin{tikzcd} \widetilde{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}G \arrow[d] \\ QX \arrow[r, "g"] & \mathbf{B}G \end{tikzcd}$$

Remark 5.16. In the situation above, $\widetilde{P}$ is a G-principal ∞-bundle, and the construction of taking this pullback is precisely the map Rec described in [NSS14b, Definition 3.93].

Thus $\widetilde{P}$ is a simplicial presheaf, or equivalently a simplicial diffeological space such that

$$\widetilde{P}_1 = \coprod_{U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_1} \times_G (G \times G), \qquad \widetilde{P}_0 = \coprod U_{p_0} \times G.$$

We can visualize the elements of $\widetilde{P}_0$ as points labeled by (p,u,h), where $p : U_p \to X$ is a plot, $u \in U_p$, and $h \in G$. A map (i.e. element of $\widetilde{P}_1$) from (p,u,h) to (q,v,k) is a map

$f : U_p \to V_q$ of plots, such that $f(u) = v$ and $k = g_f(u)h$. Now consider the relation $(p,u,h) \sim (q,v,k)$ where two points are equivalent if there exists a morphism between them. This relation is reflexive and transitive, but it is not a symmetric, so consider the smallest equivalence relation containing this relation, namely its symmetric closure. Namely we consider $(p,u,h) \sim (q,v,k)$ if there is a zig-zag of morphisms that connect the two. We can quotient $\widetilde{P_0}$ by this equivalence relation, obtaining a diffeological space $P \cong \mathrm{coeq}\left(\widetilde{P_1} \rightrightarrows \widetilde{P_0}\right)$. There is a canonical map

$$P \xrightarrow{\pi} X, \qquad [p,u,h] \mapsto p(u) \tag{36}$$

and this is well defined by the same argument as in the proof of Lemma 4.21.

Proposition 5.17. Given a diffeological group G, diffeological space X and a map $QX \xrightarrow{g} \mathbf{B}G$, the map $P \xrightarrow{\pi} X$ defined in (36) is a diffeological principal G-bundle.

Proof. First let us show that there is an action of G on P. Let the action be defined by $[p,u,h] \cdot g = [p,u,hg]$. It is clear that this is well defined, as any zig-zag that identifies $[p,u,h]$ with $[q,v,k]$ will also identify $[p,u,hg]$ with $[q,v,kg]$. Let us show that if $p : U \to X$ is a plot, then p^*P is G-equivariantly diffeomorphic to $U \times G$. Note that

$$p^*P = \{(u,[q,v,k]) \in U \times P \,:\, p(u) = q(v)\}$$

So given a point $(u,[q,v,k]) \in p^*P$, we have a diagram

$$\begin{tikzcd} * \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, "x"] & V \arrow[d, "q"] \\ U \arrow[r, "p"'] & X \end{tikzcd}$$

Now let us define a map $\varphi : p^*P \to U \times G$ as follows. Note that $[q,v,k] = [x,*,g_v^{-1}(*)k]$, as $v : * \to V$ is a map of plots. Similarly $[p,u,g_u(*)g_v^{-1}(*)k] = [x,*,g_v^{-1}(*)k]$. For brevity, if $u : * \to U$ is a constant map of plots, then denote $g_u(*)$ by g_u. So define $\varphi(u,[q,v,k]) = (u, g_u g_v^{-1} k)$. Let us show that this is well defined. If $[r,w,l] = [q,v,k]$, then there exists a zig-zag of morphisms connecting (r,w,l) and (q,v,k). Suppose that there is a single morphism between them, namely there is a map $f : V \to W$ such that the following diagram commutes

$$\begin{tikzcd} * \arrow[r, "w"] \arrow[d, "v"'] & W \arrow[d, "r"] \\ V \arrow[r, "q"'] \arrow[ur, "f"] & X \end{tikzcd}$$

and $l = g_f(v)k$. Notice that $g_w = g_{fv} = (g_f \circ v)g_v$ by the cocycle condition (33). Thus $g_w = g_f(v)g_v$, and therefore $g_v^{-1} = g_w^{-1} g_f(v)$. Thus

$$\varphi(u,[r,w,l]) = (u, g_u g_w^{-1} l) = (u, g_u g_w^{-1} g_f(v)k) = (u, g_u g_v^{-1} k) = \varphi(u,[q,v,k]).$$

If (r,w,l) and (q,v,k) are connected by a zig-zag of morphisms, then using the above argument to every morphism in the zig-zag shows that $\varphi(u,[r,w,l]) = \varphi(u,[q,v,k])$. So φ is well defined. It is also G-equivariant, as $\varphi(u,[q,v,k]\cdot g) = \varphi(u,[q,v,kg]) = (u, g_u g_v^{-1} kg) = (u, g_u g_v^{-1} k) \cdot g$. We define an inverse $\psi : U \times G \to p^*P$ by $(u,g) \mapsto (u,[p,u,g])$. This is clearly G-equivariant, and it is easy to see that $\varphi\psi = 1_{U\times G}$ and $\psi\varphi = 1_{p^*P}$. Thus P is plotwise trivial.

Now let us show that G acts on the fibers of π freely and transitively. However this is imediate, as a fiber of π is in particular a pullback:

$$\begin{tikzcd} \pi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\pi"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

and every constant map $* \to X$ is a plot, and we've already shown that for any plot this pullback is trivial, and thus $\pi^{-1}(x) \cong * \times G \cong G$, which acts freely and transitively on itself by right multiplication. Thus by Lemma 5.6, the action map $P \times G \to P \times_X P$ is a diffeomorphism.

Thus by Lemma 5.5, $P \xrightarrow{\pi} X$ is a diffeological principal G-bundle. $\square$

Thus we have constructed a map $\phi : \mathsf{Coc}(X,G)_0 \to \mathsf{DiffPrin}_G(X)$ on the objects of the respective categories. To every cocycle $g : QX \to \mathbf{B}G$ we can construct a diffeological principal G-bundle as above.

Now given a diffeological principal G-bundle $\pi : P \to X$, suppose that we have a map of plots $f_0 : U_{p_1} \to U_{p_0}$ over X. Then pulling back P along these maps gives a map $f_0^* : U_{p_1} \times G \to U_{p_0} \times G$ over X. This map is given by $(x_{p_1}, h) \mapsto (f_0(x_{p_1}), g_{f_0}(x_{p_1})h)$. If we have a composition of maps of plots over X such as $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$, then we similarly get a map $(f_0f_1)^*$ defined by

$$(f_0f_1)^*(x_{p_2}, h) = (f_0f_1(x_{p_2}), g_{f_0f_1}(x_{p_2})h).$$

However

$$f_1^*f_0^*(x_{p_2}, h) = (f_0f_1(x_{p_2}), g_{f_0}(f_1(x_{p_2}))g_{f_1}(x_{p_2})h).$$

Since these expressions are equal we recover the cocycle condition (33) and thus a map $g : QX \to BG$. Thus we have obtained a map $\psi : \mathsf{DiffPrin}_G(X)_0 \to \mathsf{Coc}(X,G)_0$.

Proposition 5.18. Suppose we have two maps $g, g' : QX \to \mathbf{B}G$, then a 1-simplex $h \in \mathsf{Tot}(\mathbf{B}G(QX))_1$ between them defines a G-equivariant map of the resulting diffeological principal G-bundles $h : \phi(g) \to \phi(g')$.

Proof. Suppose $P = \phi(g)$ and $P' = \phi(g')$. Then we define a map $h : P \to P'$ as follows. For $[p,u,k] \in P$, let $h([p,u,k]) = [p,u,h_p(u)k]$. Let us show that this map is well defined. Suppose $[q,v,l] = [p,u,k]$, namely there is a zig-zag of morphisms connecting (p,u,k) to (q,v,l). We want to then show that $[q,v,h_q(v)l] = [p,u,h_p(u)k]$. Suppose the zig-zag is of the following form.

$$\begin{tikzcd} & * \arrow[dl, "u"'] \arrow[d, "w"] \arrow[dr, "v"] & \\ U \arrow[dr, "p"'] & W \arrow[l, "f_0"'] \arrow[r, "f_1"] \arrow[d, "r"] & V \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

namely there is a triple (r, w, m) such that $f_0 r = p$, $k = g_{f_0}(w)m$, $f_0(w) = u$, $f_1 r = q$, $l = g_{f_1}(w)m$ and $f_1(w) = v$. Thus $k = g_{f_0}(w)(g_{f_1}(w))^{-1}l$. So we have the following

$$\begin{aligned}
[p, u, h_p(u)k] &= [p, u, h_p(f_0(w))g_{f_0}(w)(g_{f_1}(w))^{-1}l] \\
&= [p, u, g'_{f_0}(w)h_r(w_0)(g_{f_1}(w))^{-1}l] \\
&= [p, u, g'_{f_0}(w)(g'_{f_1}(w))^{-1}h_q(f_1(w))l],
\end{aligned}$$

where the second and third equalities follow from (34). Now since f_0 and f_1 are maps of plots it follows that

$$\begin{aligned}
[p, u, g'_{f_0}(w)(g'_{f_1}(w))^{-1}h_q(f_1(w))l] &= [r, w, (g'_{f_1}(w))^{-1}h_q(v)l] \\
&= [q, v, h_q(v)l],
\end{aligned}$$

which was precisely what we wished to show. It is easy to see that this argument generalizes to all zig-zags, and therefore shows that $h : P \to P'$ is well defined. Since $[p, u, k] \cdot g = [p, u, kg]$, it is easy to see that h is G-equivariant, and since h is built from smooth operations it is smooth. $\square$

If one has a map $h : P \to P'$ of diffeological principal G-bundles, then pulling back h along a plot $p : U \to X$ provides by the universal property of pullbacks, a unique G-equivariant map $h_p : U \times G \to U \times G$ over U making the following diagram commute.

$$\begin{tikzcd}
 & & U \times G \arrow[d] \arrow[drr] \arrow[dr, phantom, "\lrcorner", very near start] & & \\
U \times G \arrow[urr, "\widetilde{h}_p"] \arrow[d] \arrow[drr] \arrow[dr, phantom, "\lrcorner", very near start] & & U \arrow[drr] & & P' \arrow[d] \\
U \arrow[urr, equal] \arrow[drr, "p"'] & & P \arrow[urr, "h"] \arrow[d] & & X \arrow[dll, equal] \\
 & & X & &
\end{tikzcd}$$

Since the map $\widetilde{h}_p$ is G-equivariant and over U, it can be identified with a map $h_p : U \to G$. Now we wish to show that this defines a morphism of cocycles $h : \phi(P) \to \phi(P')$. So suppose $f : U \to V$ is a map of plots of X, then we have the following commutative diagram

$$\begin{tikzcd}
 & & U\times G \arrow[d] \arrow[drr, "f^*"] & & & & \\
U\times G \arrow[urr, "\widetilde{h}_p"] \arrow[d] \arrow[drr, "f^*"] & & U \arrow[drr, "f"] & & V\times G \arrow[d] \arrow[drr] & & \\
U \arrow[urr, equal] \arrow[drr, "f"] & & V\times G \arrow[urr, "\widetilde{h}_q"] \arrow[d] \arrow[drr] & & V \arrow[drr] & & P' \arrow[d] \\
 & & V \arrow[urr, equal] \arrow[drr, "q"] & & P \arrow[urr, "h"] \arrow[d] & & X \\
 & & & & X \arrow[urr, equal] & &
\end{tikzcd}$$

and tracing an element along the top left horizontal square gives us

$$f^*\widetilde{h}_p(u,g) = f^*(u,h_p(u)g) = (v,g'_f(u)h_p(u)g) = (v,h_q(v)g_f(u)g) = \widetilde{h}_q(v,g_f(u)g) = \widetilde{h}_q f^*(u,g).$$

Which implies that $g'_f \cdot h_p = (h_q \circ f)\cdot g_f$, so the h_p form a morphism of cocycles.

It is not hard to see that we've constructed functors $\phi : \mathsf{Coc}(X,G) \to \mathsf{DiffPrin}_G(X)$ and $\psi : \mathsf{DiffPrin}_G(X) \to \mathsf{Coc}(X,G)$, which send cocycles and morphisms of cocycles to bundles and morphisms of bundles and vice versa, respectively. These functors define an equivalence of categories. Indeed, suppose $\pi : P \to X$ is a diffeological principal G-bundle, we want to show that there is an isomorphism $\phi\psi(P) \to P$. If $[p,u,h] \in \phi\psi(P)$, then consider the plotwise trivialization

$$\begin{tikzcd}
U\times G \arrow[r, "\varphi_p"] \arrow[dr] & p^*P \arrow[r, "i_p"] \arrow[d] & P \arrow[d, "\pi"] \\
 & U \arrow[r, "p"'] & X
\end{tikzcd}$$

and consider the map $i_p\varphi_p : U\times G \to P$. If there is a plot map between two plots p and q, then we have the following commutative diagram

$$\begin{tikzcd}
U\times G \arrow[r, "\varphi_p"] \arrow[dr] \arrow[ddr, "g_f"'] & p^*P \arrow[d] \arrow[drr, "i_p"] \arrow[ddr, "f^*"] & & \\
 & U \arrow[drr, "p"] \arrow[ddr, "f"] & & P \arrow[d] \\
 & V\times G \arrow[r, "\varphi_q"] \arrow[dr] & q^*P \arrow[d] \arrow[ur, "i_q"] & X \\
 & & V \arrow[ur, "q"'] &
\end{tikzcd} \tag{37}$$

which shows that $i_p\varphi_p(u,h) = i_q\varphi_q(f(u),g_f(u)h)$. This extends to all zig-zags. Thus the map $[p,u,h] \mapsto i_p\varphi_p(u,h)$ is well defined as a map $\phi\psi(P) \to P$. It is not hard to see that this map is smooth and G-equivariant, and therefore is a map of diffeological principal

G-bundles, which is therefore an isomorphism. Similarly if g is a cocycle of X with values in G, then a diagram chase shows that $\psi\phi(g) = g$. This proves the following result.

Proposition 5.19. The functors ϕ and ψ define an equivalence of categories
$$\mathsf{Coc}(X, G) \simeq \mathsf{DiffPrin}_G(X).$$

From this our main result follows.

Theorem 5.20. The nerve of the category of diffeological principal G-bundles on X and the nerve of the category of G-principal ∞-bundles on X are weak homotopy equivalent

$$N\mathsf{Prin}_G^\infty(X) \simeq N\mathsf{DiffPrin}_G(X). \tag{38}$$

Proof. Proposition 5.19 implies that there is a homotopy equivalence of simplicial sets $N\mathsf{Coc}(X, G) \simeq N\mathsf{DiffPrin}_G(X)$. Combining this with Lemma 5.14 and Proposition 5.11 gives the result. □

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