THE DIFFEOLOGICAL ČECH-DE RHAM OBSTRUCTION

EMILIO MINICHIELLO

arXiv:2401.09400v1 [math.DG] 17 Jan 2024

ABSTRACT. Using higher topos theory, we explore the obstruction to the Čech-de Rham map being an isomorphism in each degree for diffeological spaces. In degree 1, we obtain an exact sequence which interprets Iglesias-Zemmour's construction [Igl23] in ∞-stack cohomology. We obtain new exact sequences in all higher degrees. These exact sequences are constructed using homotopy pullback diagrams that include the ∞-stack classifying higher $\mathbb{R}$-bundle gerbes with connection. We also obtain a conceptual and succinct proof that the ∞-stack cohomology of the irrational torus T_K for $K \subset \mathbb{R}$ a diffeologically discrete subgroup, agrees with the group cohomology of K with values in $\mathbb{R}$. Finally, for a Lie group G, we prove that the groupoid of diffeological principal G-bundles with connection one obtains via higher topos theory is equivalent to the groupoid of diffeological principal G-bundles with connection defined in [Wal12].

CONTENTS

1. INTRODUCTION

Classical differential geometry involves the study of finite dimensional smooth manifolds. As a theory, it has many achievements. One of its most celebrated is the **Čech-de Rham Theorem**, more commonly known as the de Rham Theorem[1]. The Čech-de Rham Theorem, proven in 1931 by de Rham [deR31], states that if M is a finite dimensional smooth manifold, then there is an isomorphism

$$H^k_{\mathrm{dR}}(M) \cong \check{H}^k(M, \mathbb{R}^\delta), \tag{1}$$

where $H^k_{\mathrm{dR}}(M)$ denotes the de Rham cohomology of M, and $\check{H}^k(M, \mathbb{R}^\delta)$ denotes the Čech cohomology of M with values in $\mathbb{R}^\delta$, the constant sheaf on the discrete group of real numbers. There are many good textbook accounts of the Čech-de Rham Theorem, such as [BT+82, Chapter II] and [GQ22, Chapter 9]. The de Rham cohomology of a finite dimensional smooth manifold M is constructed using its smooth structure, but the Čech-de Rham Theorem shows that the de Rham cohomology of M is independent of this smooth structure and depends only on the topology of M.

CUNY GRADUATE CENTER, EMAIL ADDRESS: EMINICHIELLO@GRADCENTER.CUNY.EDU

[1] We call it the Čech-de Rham Theorem because some authors use "the de Rham Theorem" to refer to the isomorphism between de Rham cohomology and singular cohomology.

Diffeology is a modern framework for differential geometry whose main objects of study are diffeological spaces, encompassing smooth manifolds, orbifolds, and mapping spaces. The category of diffeological spaces is better behaved than the category of finite dimensional smooth manifolds, indeed the category of diffeological spaces is complete, cocomplete and cartesian closed [Igl13]. This makes diffeological spaces attractive to geometers who study spaces that are not finite dimensional smooth manifolds. However, this generalization comes at the cost of losing many of the theorems and constructions of classical differential geometry[2]. Much contemporary work has gone into extending these constructions and theorems to diffeological spaces. The textbook [Igl13] by Iglesias-Zemmour has in particular pushed the theory quite far, defining differential forms, de Rham cohomology, singular cohomology, fiber bundles, and smooth homotopy groups of diffeological spaces amongst many other contributions.

In [Igl88], Patrick Iglesias-Zemmour proved that the Čech-de Rham Theorem does not hold in general for diffeological spaces. Interestingly, this result was written as a preprint in French in the late 80s and was only recently published in English as [Igl23]. Furthermore Iglesias-Zemmour obtained an exact sequence

$$0 \to H^1_{\mathrm{dR}}(X) \to \check{H}^1_{PIZ}(X, \mathbb{R}^\delta) \to {}^dE_2^{1,0}(X) \xrightarrow{c_1} H^2_{\mathrm{dR}}(X) \to \check{H}^2_{PIZ}(X, \mathbb{R}^\delta)$$

which is a receptacle for the obstruction to the Čech-de Rham Theorem. The group ${}^dE_2^{1,0}(X)$ is the subgroup of the group of isomorphism classes of diffeological principal $\mathbb{R}$-bundles that admit a connection. If this group is trivial, as it is for all finite dimensional smooth manifolds, then $H^1_{\mathrm{dR}}(X) \cong \check{H}^1_{PIZ}(X)$. However, the situation for higher degrees is not addressed in [Igl23]. Iglesias-Zemmour writes "We must acknowledge that the geometrical natures of the higher obstructions of the De Rham theorem still remain uninterpreted. It would be certainly interesting to pursue this matter further" [Igl23, Page 2]. In this paper, we obtain such an interpretation of the higher obstructions.

In [Min22], we introduced a generalization of Čech cohomology for diffeological spaces that we call **∞-stack cohomology**. If X is a diffeological space and A is a diffeological abelian group, then $\check{H}^k_\infty(X, A)$ denotes the kth ∞-stack cohomology of X with values in A.

Currently, there are four definitions of Čech cohomology for diffeological spaces in the literature. They are Iglesias-Zemmour's cohomology from [Igl23], which we call PIZ cohomology, there is ∞-stack cohomology [Min22], there is Krepski-Watts-Wolbert cohomology [KWW21] and there is Ahmadi's cohomology [Ahm23]. In [Min22, Section 5.3], the first three Čech cohomologies were compared, and some relationships deduced, but it is currently unknown if any of the above cohomology theories agree in general.

This paper is a sequel to [Min22], where we explored the connection between diffeological spaces and higher topos theory to study diffeological principal G-bundles. When G is a diffeological group, not necessarily abelian, it is still possible to define ∞-stack cohomology in degree 1, $\check{H}^1_\infty(X, G)$. We proved [Min22, Corollary 6.9] that degree 1 ∞-stack cohomology is in bijection with isomorphism classes of diffeological principal G-bundles over X. In fact, we obtained a much stronger result by showing that the nerve of the groupoid of diffeological principal G-bundles is weak equivalent to the nerve of the category of G-principal ∞-bundles on X [Min22, Theorem 6.8].

In this paper, we study two cases where the tools of higher topos theory help us better understand diffeological spaces. The first case is studying the ∞-stack cohomology of the irrational torus. The irrational torus was the first example of a nontrivial diffeological space with trivial underlying topology, see [Igl20]. In [Igl23], Iglesias-Zemmour proved that if $K \subset \mathbb{R}$ is a diffeologically discrete subgroup, then the PIZ cohomology of the irrational torus $T_K = \mathbb{R}/K$ is isomorphic to the group cohomology of K with values in $\mathbb{R}$. However, his proof of this, [Igl23, Section II], is computational. In Section 4, we prove

[2]Many of these theorems are lost because not all diffeological spaces have partitions of unity, a crucial ingredient to many theorems in differential geometry.

Theorem 4.4. There is an isomorphism

$$\check{H}^n_\infty(T_K, \mathbb{R}^\delta) \cong H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta)$$

of abelian groups, for every $n \geq 0$, where $\mathbb{R}^\delta$ denotes the discrete group of real numbers, and where $H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta)$ denotes the group cohomology of K with coefficients in $\mathbb{R}^\delta$.

Theorem 4.4 supports the conjecture that PIZ cohomology and ∞-stack cohomology agree. The proof of Theorem 4.4 is short and conceptual. It uses the shape functor $\int$, much beloved by higher differential geometers [BBP22], [Bun22], [Sch13], [Clo23], [Car15], in a crucial way, reducing the ∞-stack cohomology of T_K to the singular cohomology of the classifying space $\mathbf{B}K$. This demonstrates the advantage of using ∞-stack cohomology to study diffeological spaces.

The second case, which makes up the bulk of the paper, is to use ∞-stack cohomology, and more generally the framework of higher topos theory, to study the diffeological Čech-de Rham obstruction. First we obtain a homotopy pullback diagram of ∞-stacks.

Theorem 7.1. For every $k \geq 1$, there exists a commutative diagram of ∞-stacks of the following form

(2)
$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathbf{B}^k\mathbb{R}^\delta \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \mathbf{B}^k_\nabla\mathbb{R} \arrow[r] \arrow[d] & \Omega^{k+1}_{\mathrm{cl}} \arrow[r] \arrow[d] & \Omega^{k+1} \arrow[d] \\
 & \mathbf{B}^k\mathbb{R} \arrow[r] \arrow[d] & \mathbf{B}^k\Omega^1_{\mathrm{cl}} \arrow[r] \arrow[d] & \Omega^{1\leq\bullet\leq k+1} \arrow[d] \\
 & * \arrow[r] & \mathbf{B}^{k+1}\mathbb{R}^\delta \arrow[r] & \mathbf{B}^{k+1}_\nabla\mathbb{R}
\end{tikzcd}$$

furthermore every commutative square in this diagram is a homotopy pullback square in the Čech model structure on simplicial presheaves over Cart.

Such diagrams are often used in higher category-theoretic treatments of differential cohomology, see [Sch13], [ADH21], [Jaz21]. One can think of an ∞-stack as a classifying object for a mathematical structure, such as diffeological principal G-bundles, as in [Min22]. Thus the above diagrams can be thought of as tight relationships between the corresponding mathematical structures.

Of particular interest is the ∞-stack $\mathbf{B}^k_\nabla\mathbb{R}$. This is the ∞-stack which classifies diffeological **$\mathbb{R}$-bundle $(k-1)$-gerbes with connection**. Cohomology with values in this ∞-stack is called the kth pure differential cohomology in [Jaz21]. From Theorem 7.1 we are immediately able to obtain the following result.

Corollary 7.2. For every diffeological space X, there is an exact sequence of vector spaces

(3)
$$0 \to \check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^k_{\infty,\nabla}(X, \mathbb{R}) \to \Omega^{k+1}_{\mathrm{cl}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta).$$

Near the completion of this paper, we learned that an analogous exact sequence was also obtained in [Jaz21, Page 27] using completely different methods in the framework of homotopy type theory. The above exact sequence allows us to compute the pure differential cohomology of the irrational torus.

Theorem 7.3. Let T_α denote the irrational torus, then

(4)
$$\check{H}^k_{\infty,\nabla}(T_\alpha, R) \cong \begin{cases} \mathbb{R}^2, & k = 1, \\ \mathbb{R}, & k = 2, \\ 0, & k > 2. \end{cases}$$

While Corollary 7.2 is useful for computations with the irrational torus, it is desirable to have an exact sequence including de Rham cohomology rather than closed forms. This is obtained in the following result.

Theorem 7.5. Given a diffeological space X and $k \geq 1$, the sequence of vector spaces

$$\check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^k_{\mathrm{conn}}(X, \mathbb{R}) \to H^{k+1}_{\mathrm{dR}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta)$$

is exact.

When $k = 1$, we obtain an additional piece to this exact sequence.

Theorem 7.7. Given a diffeological space X, the sequence of vector spaces

$$0 \to H^1_{\mathrm{dR}}(X) \to \check{H}^1_\infty(X, \mathbb{R}^\delta) \to \check{H}^1_{\mathrm{conn}}(X, \mathbb{R}) \to H^2_{\mathrm{dR}}(X) \to \check{H}^2_\infty(X, \mathbb{R}^\delta) \tag{5}$$

is exact.

The above exact sequence is exactly analogous to the exact sequence obtained by Iglesias-Zemmour in [Igl23].

In Appendix A we turn to the study of connections for diffeological principal bundles. This theory is still in its infancy, and there are a few references that give varying definitions of diffeological connections [Igl13, Section 8.32], [Wal12, Section 3], [MW17, Section 4]. The theory of ∞-stacks provides another definition. Let G be a Lie group, and U a cartesian space. Then let $\Omega^1(U, \mathfrak{g})//G$ denote the groupoid whose objects are differential 1-forms $\omega \in \Omega^1(U, \mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G, and where there is a morphism $g : \omega \to \omega'$ if there exists a smooth map $g : U \to G$ such that

$$\omega' = \mathrm{Ad}_g^{-1}(\omega) + g^* \mathrm{mc}(G)$$

where $\mathrm{mc}(G)$ denotes the Maurer-Cartan form of G. Taking the nerve of this groupoid, and letting U vary defines an ∞-stack $\Omega^1(-, \mathfrak{g})//G$, which amongst others has been studied in [FSS+12], [FH13]. We connect this notion of connection to that given in [Wal12, Definition 3.2.1] in the following result.

Theorem A.3. Given a diffeological space X and a Lie group G, the functor

$$\mathsf{Cons}_\nabla : \mathsf{Coc}_\nabla(X, G) \to \mathsf{Wal}_G(X), \tag{6}$$

is an equivalence of groupoids, where $\mathsf{Coc}_\nabla(X, G)$ is the groupoid whose objects are maps $QX \to \Omega^1(-, \mathfrak{g})//G$, where QX is a cofibrant replacement of X in the projective model structure on simplicial presheaves, and $\mathsf{Wal}_G(X)$ is the groupoid of diffeological principal G-bundles with connection as defined in [Wal12, Definition 3.2.1].

To compute ∞-stack cohomology, one needs a workable model of the derived mapping space $\mathbb{R}\mathrm{Hom}(X, A)$, when X is a diffeological space and A is a presheaf of chain complexes. In Appendix B, we obtain such a model, which reduces many computations with ∞-stacks to manipulations with double complexes. As a corollary, we obtain a simple and direct proof of the following well known folklore result.

Proposition B.4. Let C be a cosimplicial chain complex, then

$$\mathrm{holim}_{n \in \Delta} C^n \simeq \mathrm{tot}\, C, \tag{7}$$

where we are computing the homotopy limit in the category of chain complexes equipped with the projective model structure, and $\mathrm{tot}\, C$ denotes the total complex of C.

The paper is organized as follows. In Section 2, we introduce diffeological spaces and place them in the context of sheaf theory. In Section 3, we introduce simplicial presheaves, show how diffeological spaces embed into simplicial presheaves, and introduce the shape functor. In Section 4, we prove that the ∞-stack cohomology of the irrational torus T_K is isomorphic to the group cohomology of K with values in $\mathbb{R}$. In Section 5, we introduce the Dold-Kan correspondence, which is a core tool we use for the rest of the paper. In Section 6, we introduce the main ∞-stacks that will be used in the paper, and compute various examples of ∞-stack cohomology. In Section 7, we prove the main results of this paper, Theorem 7.5 and Theorem 7.7. In Appendix A, we prove that our notion of diffeological principal G-bundles with connection using ∞-stacks agrees with Waldorf's [Wal12]. In Appendix B, we prove a technical result allowing us to easily compute ∞-stack cohomology when the coefficient ∞-stack comes from a presheaf of chain complexes. In Appendix C we prove Theorem 7.1.

2. Smooth Sheaves and Diffeological Spaces

In this section we briefly describe diffeological spaces and their connection to sheaves on Cart. See [Min22, Section 2] for more details.

Definition 2.1. Let M be a finite dimensional smooth manifold[3]. We say a collection of subsets $\mathcal{U} = \{U_i \subseteq M\}_{i\in I}$ is an **open cover** if each U_i is an open subset of M, and $\bigcup_{i\in I} U_i = M$. If U is a finite dimensional smooth manifold diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, we call U a **cartesian space**. We call $\mathcal{U} = \{U_i \subseteq M\}$ a **cartesian open cover** of a manifold M if it is an open cover of M and every U_i is a cartesian space. We say that $\mathcal{U}$ is a **good open cover** if it is a cartesian open cover, and further every finite non-empty intersection $U_{i_0\dots i_k} = U_{i_0} \cap \dots \cap U_{i_k}$ is a cartesian space.

Let Man denote the category whose objects are finite dimensional smooth manifolds and whose morphisms are smooth maps. Let Cart denote the full subcategory whose objects are cartesian spaces. Given a set X, let $\mathsf{Param}(X)$ denote the set of **parametrizations** of X, namely the collection of set functions $p : U \to X$, where $U \in \mathsf{Cart}$.

Definition 2.2. A **diffeology** on a set X, consists of a collection $\mathcal{D}$ of parametrizations $p : U \to X$ satisfying the following three axioms:

(1) $\mathcal{D}$ contains all points $\mathbb{R}^0 \to X$,
(2) If $p : U \to X$ belongs to $\mathcal{D}$, and $f : V \to U$ is a smooth map, then $pf : V \to X$ belongs to $\mathcal{D}$, and
(3) If $\{U_i \subseteq U\}_{i\in I}$ is a good open cover of a cartesian space U, and $p : U \to X$ is a parametrization such that $p|_{U_i} : U_i \to X$ belongs to $\mathcal{D}$ for every $i \in I$, then $p \in \mathcal{D}$.

A set X equipped with a diffeology $\mathcal{D}$ is called a **diffeological space**. Parametrizations that belong to a diffeology are called **plots**. We say a set function $f : X \to Y$ between diffeological spaces is **smooth** if for every plot $p : U \to X$ in $\mathcal{D}_X$, the composition $pf : U \to Y$ belongs to $\mathcal{D}_Y$. We often denote the set of smooth maps from X to Y by $C^\infty(X, Y)$. Let Diff denote the category of diffeological spaces.

Every manifold M is canonically a diffeological space by considering the set of parametrizations $p : U \to M$ that are smooth in the classical sense. This gives a diffeology on M, called the **manifold diffeology**. One can show [Igl13, Chapter 4] that the manifold diffeology defines a fully faithful functor $\mathsf{Man} \hookrightarrow \mathsf{Diff}$.

Diffeology extends many constructions and concepts from classical differential geometry to diffeological spaces, such as the theory of bundles.

Definition 2.3. We say that a map $\pi : X \to Y$ of diffeological spaces is a **subduction** if it is surjective, and for every plot $p : U \to Y$, there exists a good open cover $\{U_i \subseteq U\}$, and plots $p_i : U_i \to X$ making the following diagram commute

$$\begin{tikzcd} U_i \arrow[r, "p_i"] \arrow[d, hook] & X \arrow[d, "\pi"] \\ U \arrow[r, "p"'] & Y \end{tikzcd} \tag{8}$$

Definition 2.4. A **diffeological group** is a group G equipped with a diffeology such that the multiplication map $m : G\times G \to G$, and inverse map $i : G \to G$ are smooth. A right **diffeological group action** of a diffeological group G on a diffeological space X is a smooth map $\rho : X\times G \to X$ such that $\rho(x, e_G) = x$, and $\rho(\rho(x, g), h) = \rho(x, gh)$, where e_G denotes the identity element of G.

[3]We will assume throughout this paper that manifolds are Hausdorff and paracompact.

Definition 2.5. Let G be a diffeological group, and P be a diffeological right G-space. A map $\pi : P \to X$ of diffeological spaces is a **diffeological principal G-bundle** if:

(1) the map $\pi : P \to X$ is a subduction, and
(2) the map $\mathrm{act} : P \times G \to P \times_X P$ defined by $(p, g) \mapsto (p, p \cdot g)$, which we call the **action map** is a diffeomorphism.

A map of diffeological principal G-bundles $P \to P'$ over X is a diagram

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

where f is a G-equivariant smooth map. A diffeological principal G-bundle P is said to be **trivial** if there exists an isomorphism $\varphi : X \times G \to P$, called a **trivialization**, where $\mathrm{pr}_1 : X \times G \to X$ is the product bundle. Let $\mathsf{DiffPrin}_G(X)$ denote the category of diffeological principal G-bundles over a diffeological space X.

In [Min22], we proved that diffeological principal bundles can be classified using cocycles in a way reminiscent of classical differential geometry. However, rather than using cocycles defined over an open cover, we use cocycles defined on plots. Let $\mathsf{Plot}(X)$ denote the category whose objects are plots $p : U \to X$ of X and whose morphisms $f : p \to p'$ are smooth maps $f : U \to U'$ such that $p'f = p$.

Definition 2.6. Given a diffeological space X and a diffeological group G, call a collection $g = \{g_{f_0}\}$ of smooth maps $g_{f_0} : U_{p_1} \to G$ indexed by maps of plots $f_0 : U_{p_1} \to U_{p_0}$ of X a **G-cocycle** if for every pair of composable plot maps of X

$$U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$$

it follows that

$$g_{f_0 f_1} = (g_{f_0} \circ f_1) \cdot g_{f_1}. \tag{9}$$

We call (9) the **diffeological G-cocycle condition**.

Given two G-cocycles, g, g', we say a collection $h = \{h_{p_0}\}$ of smooth maps $h_{p_0} : U_{p_0} \to G$ indexed by plots of X is a **morphism of G-cocycles** $h : g \to g'$ if for every map $f_0 : U_{p_1} \to U_{p_0}$ of plots of X, it follows that

$$g'_{f_0} \cdot h_{p_1} = (h_{p_0} \circ f_0) \cdot g_{f_0}. \tag{10}$$

Given a diffeological space X and a G-cocycle g on X, we can construct a diffeological principal G-bundle $\pi : P \to X$, by taking the quotient

$$P = \left(\coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \times G \right) /\sim \tag{11}$$

where $\sim$ is the smallest equivalence relation such that $(x_{p_1}, k_1) \sim (x_{p_0}, k_0)$ if there exists a map $f_0 : U_{p_1} \to U_{p_0}$ of plots such that $f_0(x_{p_1}) = x_{p_0}$ and $k_0 = g_{f_0}(x_{p_1}) \cdot k_1$. We let $\pi = \mathsf{Cons}(g)$, short for construction. In fact, this construction defines a functor from the category $\mathsf{Coc}(X, G)$ of G-cocycles on X to the category of diffeological principal G-bundles.

Theorem 2.7 ([Min22, Theorem 3.15])**.** *Given a diffeological space X and a diffeological group G, the functor*

$$\mathsf{Cons} : \mathsf{Coc}(X, G) \to \mathsf{DiffPrin}_G(X) \tag{12}$$

is an equivalence of groupoids.

While extending the classical theory, there are constructions one can do with diffeological spaces that are not available to smooth manifolds:

(1) Given a diffeological space X, and a subset $A \overset{i}{\hookrightarrow} X$ a subset. Then consider the set of parametrizations $p : U \to A$ such that $ip : U \to X$ is a plot of X. This collection is a diffeology, called the **subspace diffeology** on A,

(2) Given a diffeological space X and an equivalence relation $\sim$ on X, let $\pi : X \to X/\sim$ denote the resulting quotient function on sets. Consider the set of parametrizations $p : U \to X/\sim$ such that there exists a good open cover $\{U_i \subseteq U\}$ and plots $p_i : U_i \to X$ making the following diagram commute

$$\begin{tikzcd} U_i \arrow[r, "p_i"] \arrow[d, hook] & X \arrow[d, "\pi"] \\ U \arrow[r, "p"'] & X/\sim \end{tikzcd}$$

This forms a diffeology on $X/\sim$, called the **quotient diffeology**,

(3) Given a pair X and Y of diffeological spaces, the set of parametrizations $p : U \to X \times Y$ such that the composites $\pi_1 \circ p$ and $\pi_2 \circ p$ are plots of X and Y respectively, forms a diffeology, called the **product diffeology**,

(4) Given diffeological spaces X and Y, the set of parametrizations $p : U \to C^\infty(X, Y)$ such that the transposed function $p^\# : U \times X \to Y$ is a smooth map is a diffeology, called the **functional diffeology**.

These constructions make the category of diffeological spaces considerably better than the category of finite dimensional smooth manifolds, as shown in Corollary 2.10.

Diffeological spaces inherit this nice structure from the category of smooth sheaves.

Definition 2.8. We briefly recall the relevant definitions for sheaf theory.

- A **collection of families** j on a category $\mathcal{C}$ consists of a set $j(U)$ for each $U \in \mathcal{C}$, whose elements $\{r_i : U_i \to U\} \in j(U)$ are families of morphisms over U. We call a collection of families j on $\mathcal{C}$ a **coverage** if it satisfies the following property: for every $\{r_i : U_i \to U\} \in j(U)$, and every map $g : V \to U$ in $\mathcal{C}$, then there exists a family $\{t_j : V_j \to V\} \in j(V)$ such that gt_j factors through some r_i. Namely for every t_j there exists some i and some map $s_j : V_j \to U_i$ making the following diagram commute:

$$\begin{tikzcd} V_j \arrow[r, "s_j"] \arrow[d, "t_j"'] & U_i \arrow[d, "r_i"] \\ V \arrow[r, "g"'] & U \end{tikzcd} \tag{13}$$

  The families $\{r_i : U_i \to U\} \in j(U)$ are called **covering families** over U. If a map $r_i : U_i \to U$ belongs to a covering family $r \in j(U)$, then we say that r_i is a **covering map**. If $\mathcal{C}$ is a category, and j is a coverage on $\mathcal{C}$, then we call the pair $(\mathcal{C}, j)$ a **site**.
- A **presheaf** on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. A morphism of presheaves is a natural transformation. An element $x \in F(U)$ for an object $U \in \mathcal{C}$ is called a **section** over U. If $f : U \to V$ is a map in $\mathcal{C}$, and $x \in F(V)$, then we sometimes denote $F(f)(x)$ by $x|_U$. Let $\mathrm{Pre}(\mathcal{C})$ denote the category of presheaves on $\mathcal{C}$.
- If $\{r_i : U_i \to U\}_{i \in I}$ is a covering family, then a **matching family** is a collection $\{x_i\}_{i \in I}$, $x_i \in F(U_i)$, such that given a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "f"'] & U_j \arrow[d, "r_j"] \\ U_i \arrow[r, "r_i"'] & U \end{tikzcd}$$

  then $F(f)(x_i) = F(g)(x_j)$ for all $i, j \in I$. An **amalgamation** x for a matching family $\{x_i\}$ is a section $x \in F(U)$ such that $x_i|_U = x$ for all i.

- Given a family of morphisms $r = \{r_i : U_i \to U\}$ in a category $\mathcal{C}$, we say that a presheaf $F : \mathcal{C}^{op} \to \mathsf{Set}$ is a **sheaf on** r if every matching family $\{s_i\}$ of F over r has a unique amalgamation. If j is a coverage on a category $\mathcal{C}$, we call F a **sheaf** on $(\mathcal{C}, j)$ if it is a sheaf on every covering family of j. Let $\mathsf{Sh}(\mathcal{C})$ denote the full subcategory of $\mathsf{Pre}(\mathcal{C})$ whose objects are sheaves on $(\mathcal{C}, j)$.

One can put a site structure on Cart using the coverage of good open covers, see [Min22, Section 4]. We call sheaves on Cart **smooth sheaves**. There are many interesting examples of smooth sheaves. Every cartesian space defines a representable sheaf yU. Every manifold M defines a sheaf by $U \mapsto C^\infty(U, M)$. There are also Ω^n and Ω^n_{cl} for every $n \geq 0$, the sheaves of differential n-forms and closed differential n-forms respectively. The category $\mathsf{Sh}(\mathsf{Cart})$ of smooth sheaves is "extremely nice", being a Grothendieck topos [MM12].

A sheaf X on Cart is **concrete** if $X(U)$ is a subset of the set functions $U \to X(*)$ where $*$ is the terminal object in Cart. The representable sheaves yU and the sheaves induced by manifolds M are concrete, but Ω^n and Ω^n_{cl} are not.

The full subcategory $\mathsf{ConSh}(C) \hookrightarrow \mathsf{Sh}(C)$ of concrete sheaves on a concrete site forms a quasitopos, which while not being a Grothendieck topos, is still a very "nice" category [BH11, Theorem 52].

Theorem 2.9 ([BH11, Prop 24])**.** Let Cart denote the site of cartesian spaces with the coverage of good open covers. Then there is an equivalence of categories

$$\mathsf{Diff} \simeq \mathsf{ConSh}(\mathsf{Cart}), \tag{14}$$

where $\mathsf{ConSh}(\mathsf{Cart})$ denotes the category of concrete sheaves on Cart.

Corollary 2.10. The category Diff is a quasitopos. This implies that it is a complete, cocomplete and cartesian closed category.

We refer to Theorem 2.9 as the **Baez-Hoffnung Theorem**[4]. It is the starting point of the interaction of sheaf theory and diffeology. Many aspects of the study of diffeological spaces can be restated using sheaf theory, for example a differential n-form ω on a diffeological space X as defined in [Igl13, Article 6.28] is equivalently a morphism $X \to \Omega^n$ of sheaves.

In [Min22] we took advantage of the Baez-Hoffnung Theorem to embed the category of diffeological spaces into the category of simplicial presheaves on Cart. We will delve into this idea in the next section. Once inside the category of simplicial presheaves, we can then take advantage of many homotopical tools. This in effect provides a way of obtaining a very powerful and expressive homotopy theory for diffeological spaces that subsumes the usual homotopy theory for diffeological spaces as considered in [Igl13, Chapter 5].

3. Simplicial Presheaves

In this section we detail the model categorical notions we will need for the remainder of the paper. We assume the reader is comfortable with model categories and simplicial homotopy theory, and recommend the following standard sources [Hir09], [Hov07], [GJ12], [GS06] for good references on the topics. See [Min22, Section 5] for more details.

Definition 3.1. Let $\mathsf{sPre}(\mathsf{Cart})$ denote the category whose objects are functors $\mathsf{Cart}^{op} \to \mathsf{sSet}$, which we call **simplicial presheaves**, and whose morphisms are natural transformations.

Note that $\mathsf{sPre}(\mathsf{Cart})$ is complete and cocomplete, with limits and colimits computed objectwise. There are two pairs of adjoint triples that give structure to $\mathsf{sPre}(\mathsf{Cart})$.

[4]Strictly speaking, the Baez-Hoffnung theorem gives an equivalence between the category of what we call classical diffeological spaces and the category of concrete sheaves on the site of open subsets of cartesian spaces with open covers, see [Min22, Appendix A] for a proof that this is an equivalent formulation.

(15)
$$\begin{tikzcd} \mathsf{sPre}(\mathsf{Cart}) \arrow[r, bend left, "\mathrm{colim}_{\mathsf{Cart}^{op}}"] \arrow[r, bend right, leftarrow, "\lim_{\mathsf{Cart}^{op}}"'] & \mathsf{sSet} \arrow[l, "(-)_c"'] \end{tikzcd}, \qquad \begin{tikzcd} \mathsf{sPre}(\mathsf{Cart}) \arrow[r, bend left, "\pi_0"] \arrow[r, bend right, "(-)_0"'] & \mathsf{Pre}(\mathsf{Cart}) \arrow[l, "{}^c(-)"'] \end{tikzcd}$$

where $(-)_c$ is the functor induced by restricting along the unique functor Cart $\to *$ and ${}^c(-)$ is the functor that sends a presheaf to the corresponding simplicial presheaf where all the simplicial face and degeneracy maps are the identity. We often don't use the notation ${}^c(-)$ explicitly, especially for representable presheaves, as it should be clear from context. The functors π_0 and $(-)_0$ are defined objectwise. For every $U \in$ Cart, and simplicial presheaf X on Cart, $\pi_0 X(U) = \pi_0(X(U))$, the set of connected components of $X(U)$, and $(X)_0(U) = X(U)_0$, the set of vertices of $X(U)$.

Remark 3.2. The above adjoint triples exist for any essentially small category $\mathcal{C}$ in place of Cart.

The category sPre(Cart) is tensored, cotensored and enriched over sSet. Indeed, if K is a simplicial set and X is a simplicial presheaf, then

- $X \otimes K$ is the simplicial presheaf defined objectwise by
$$(X \otimes K)(U) = (X \times K_c)(U) = X(U) \times K.$$
- X^K is the simplicial presheaf defined objectwise by
$$(X^K)(U) = X(U)^K,$$
where for simplicial sets K and L, K^L denotes the simplicial function complex.
- for any two simplicial presheaves X and Y, let $\underline{\mathsf{sPre}(\mathsf{Cart})}(X,Y)$ denote the simplicial set defined levelwise by
$$\underline{\mathsf{sPre}(\mathsf{Cart})}(X,Y)_n = \mathsf{sPre}(\mathsf{Cart})(X \otimes \Delta^n, Y).$$

This structure is compatible in the sense of the following natural isomorphisms of simplicial sets

(16)
$$\underline{\mathsf{sPre}(\mathsf{Cart})}(X \otimes K, Y) \cong \underline{\mathsf{sPre}(\mathsf{Cart})}(X, Y^K).$$

The category sPre(Cart) inherits several model structures from sSet. We say a map $f : X \to Y$ is a **projective weak equivalence** if it is an objectwise weak equivalence of simplicial sets, a **projective fibration** if it is an objectwise fibration, and a **projective cofibration** if it left lifts against all maps that are both projective weak equivalences and projective fibrations.

Theorem 3.3 ([BK72, Page 314], [Lur09, Section A.2.6])**.** The projective weak equivalences, fibrations and cofibrations define a proper, combinatorial, simplicial model category structure on sPre(Cart), called the **projective model structure** on simplicial presheaves.

Let $\mathbb{H}$ denote the category of simplicial presheaves equipped with the projective model structure. Note that [Dug01, Corollary 9.4] describes a sufficient condition on simplicial presheaves to be projective cofibrant, and it implies that all representable presheaves, denoted yU for a cartesian space U, are projective cofibrant.

Given a cartesian space U and a good cover $\mathcal{U} = \{U_i \subseteq U\}$ of U, we can form the simplicial presheaf $\check{C}(\mathcal{U})$ defined levelwise by
$$\check{C}(\mathcal{U})_n = \coprod_{i_0 \dots i_n} y(U_{i_0} \cap \dots \cap U_{i_n}).$$

We call $\check{C}(\mathcal{U})$ the **Čech nerve** of $\mathcal{U}$. There is a canonical map $\pi : \check{C}(\mathcal{U}) \to yU$. Let $\check{C}$ denote the class of morphisms $\pi : \check{C}(\mathcal{U}) \to yU$ where U ranges over the cartesian spaces and $\mathcal{U}$ ranges over the good open covers for U.

Theorem 3.4 ([DHI04, Theorem A.6])**.** The left Bousfield localization of $\mathbb{H}$ at $\check{C}$ exists. We call the resulting model structure the **Čech model structure** on sPre(Cart), and denote it by $\check{\mathbb{H}}$. It is similarly a proper, combinatorial and simplicial model category.[5]

The fibrant objects in $\check{\mathbb{H}}$ are called **∞-stacks**. They are those projective fibrant simplicial presheaves X such that the canonical map

$$\underline{\mathsf{sPre(Cart)}}(yU, X) \to \underline{\mathsf{sPre(Cart)}}(\check{C}(\mathcal{U}), X), \tag{17}$$

is a weak equivalence of simplicial sets, for every cartesian space U and good cover $\mathcal{U}$ of U. Every sheaf and classical stack of groupoids on Cart, thought of as simplicial presheaves on Cart, is an ∞-stack. See [Min22, Section 4.1] for more details.

The identity functors define a Quillen adjunction between the projective and Čech model structure on simplicial presheaves.

$$\begin{tikzcd} \mathbb{H} \arrow[r, bend left, "1_{\mathsf{sPre(Cart)}}"] \arrow[r, phantom, "\perp"] & \check{\mathbb{H}} \arrow[l, bend left, "1_{\mathsf{sPre(Cart)}}"] \end{tikzcd} \tag{18}$$

Crucially, the (easy to compute) finite homotopy limits in $\mathbb{H}$ are preserved as homotopy limits in $\check{\mathbb{H}}$, thanks to the following result.

Proposition 3.5 ([Rez10, Proposition 11.2])**.** The left Quillen functor $1_{\mathsf{sPre(Cart)}} : \mathbb{H} \to \check{\mathbb{H}}$ preserves finite homotopy limits.

Given simplicial presheaves X and Y on Cart, let Q and R denote cofibrant and fibrant replacement functors for $\check{\mathbb{H}}$ respectively, then let

$$\mathbb{R}\check{\mathbb{H}}(X, Y) = \underline{\mathsf{sPre(Cart)}}(QX, RY). \tag{19}$$

We call $\mathbb{R}\check{\mathbb{H}}(X, Y)$ the **derived mapping space** of X and Y. If X is already cofibrant, then we can take $Q = 1_{\mathsf{sPre(Cart)}}$ and if Y is already fibrant, we can take $R = 1_{\mathsf{sPre(Cart)}}$.

If X and A are simplicial presheaves, then let

$$\check{H}^0_\infty(X, A) = \pi_0 \mathbb{R}\check{\mathbb{H}}(X, A).$$

We call this the 0th ∞-stack cohomology of X with values in A.

If A is a simplicial presheaf that is objectwise a simplicial group, then we let

$$\check{H}^1_\infty(X, A) = \pi_0 \mathbb{R}\check{\mathbb{H}}(X, \overline{W}A),$$

where $\overline{W}$ is the delooping functor, see [Min22, Definition 4.29].

If A is a simplicial presheaf such that $\overline{W}^k A$, which we call its k-fold delooping, exists for $k \geq 1$, then we say that A is k-deloopable, and we let

$$\check{H}^k_\infty(X, A) = \pi_0 \mathbb{R}\check{\mathbb{H}}(X, \overline{W}^k A).$$

We call this the kth ∞-stack cohomology of X with coefficients in A. If the k-fold delooping of a simplicial presheaf A exists and is an ∞-stack, then we denote it by $\mathbf{B}^k A := \overline{W}^k A$. The following result is well known, see [Min22, Lemma 4.34] for a proof.

Lemma 3.6. If A is a presheaf of simplicial abelian groups, then A is k-deloopable for all $k \geq 1$.

There is a convenient cofibrant replacement functor for $\check{\mathbb{H}}$, given (in the notation of [Rie14, Section 4.2]) for a simplicial presheaf X by the bar construction $B(X, \mathsf{Cart}, y)$, where y denotes the Yoneda embedding $y : \mathsf{Cart} \hookrightarrow \mathsf{sPre(Cart)}$.

[5]It is important to note that the projective/objectwise weak equivalences between simplicial presheaves are still weak equivalences in the Čech model structure. Furthermore, all Čech weak equivalences between ∞-stacks are objectwise weak equivalences.

If X is a diffeological space, then by the Baez-Hoffnung Theorem (Theorem 2.9), we can consider it as a sheaf on Cart. Then ${}^cX$ is a simplicial presheaf[6]. If we apply Q to ${}^cX$ then this formula reduces to the simplicial presheaf given levelwise by

$$QX_n = \coprod_{(f_{n-1},\dots,f_0)\in N(\mathsf{Plot}(X))_n} yU_{p_n} \otimes \Delta^n. \tag{20}$$

See [Min22, Section 4.2] for more details.

Using the above cofibrant replacement functor for a diffeological space X, if A is a ∞-stack that is also objectwise a simplicial abelian group, then we can obtain an explicit description of its kth ∞-stack cohomology with values in A, given by the kth cohomology of the cochain complex obtained by taking the dual Dold-Kan correspondence functor applied to the cosimplicial abelian group

$$A(QX_0) \rightrightarrows A(QX_1) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} A(QX_2) \qquad \dots\ , \tag{21}$$

the case where A is an abelian diffeological group is given by [Min22, Corollary 4.38].

Example 3.7. Let G be a diffeological group, and consider the (strict) functor $\mathbf{B}G : \mathsf{Cart}^{op} \to \mathsf{Gpd}$ that sends a cartesian space U to the groupoid[7]

$$[C^\infty(U,G) \rightrightarrows *] \tag{22}$$

Postcomposing with the nerve functor gives us a simplicial presheaf $N\mathbf{B}G$, which we will often just denote by $\mathbf{B}G$. By [Min22, Theorem 5.17], (referencing [SS21, Lemma 3.3.29] and [Pav22a, Proposition 4.13]), $\mathbf{B}G$ is an ∞-stack.

This ∞-stack takes a central role in the theory of diffeological principal G-bundles. For every cartesian space U, there is a canonical map of groupoids

$$\mathbf{B}G(U) \to \mathsf{DiffPrin}_G(U), \tag{23}$$

that sends the point to the trivial diffeological principal G-bundle, and sends a map to G to the corresponding automorphism of the trivial bundle. This map is an equivalence of groupoids.

Furthermore, if X is a diffeological space and QX is its cofibrant replacement, then G-cocycles on X are equivalent to maps of ∞-stacks $QX \to \mathbf{B}G$ in the sense of [Min22, Lemma 6.7]. In other words, for every diffeological space X, there is a weak equivalence

$$\mathbb{R}\check{\mathbb{H}}(X, \mathbf{B}G) \simeq N\mathsf{DiffPrin}_G(X). \tag{24}$$

Thus we say that $\mathbf{B}G$ classifies diffeological principal G-bundles. This implies that

$$\check{H}^1_\infty(X,G) \cong \pi_0\mathsf{DiffPrin}_G(X), \tag{25}$$

where $\pi_0\mathsf{DiffPrin}_G(X)$ denotes the set of isomorphism classes of diffeological principal G-bundles on X.

Let us now examine the left hand side of (15). If K is a simplicial set, then K_c is the constant simplicial presheaf on K, namely $K_c(U) = K$ for all $U \in \mathsf{Cart}$. The functors making up (15) are important enough to warrant renaming. Notice that since $\mathbb{R}^0 = *$ is the terminal object in Cart, it is the initial object in Cart^{op}, thus $\lim_{U\in\mathsf{Cart}^{op}} X(U) \cong X(*)$. For $K \in \mathsf{sSet}$ and $X \in \mathsf{sPre}(\mathsf{Cart})$, we set

$$\mathrm{Disc}(K) = K_c, \qquad \Gamma(X) = \lim_{U\in\mathsf{Cart}^{op}} X(U) \cong X(*), \qquad \Pi_\infty(X) = \operatorname*{colim}_{U\in\mathsf{Cart}^{op}} X(U). \tag{26}$$

It turns out that Γ has a further right adjoint, $\mathrm{CoDisc} : \mathsf{sSet} \to \mathsf{sPre}(\mathsf{Cart})$ defined objectwise by

$$\mathrm{CoDisc}(K)(U) = K^{\Gamma(yU)}.$$

We say that $\mathrm{Disc}(K)$ is the **discrete simplicial presheaf** on K, $\Gamma(X)$ is the **global sections** of X, $\Pi_\infty(X)$ is the **fundamental ∞-groupoid** or **shape** of X, and that $\mathrm{CoDisc}(K)$ is the **codiscrete**

[6] We will often not use the notation ${}^cX$ for diffeological spaces in what follows, as it should be apparent from context what category we are considering X in.

[7] Here we use the convention discussed in [Min22, Example 5.13] for $\mathbf{B}G$.

simplicial presheaf on K. In fact, all of these adjunctions are simplicially enriched adjunctions.

Thus we obtain the following triple of simplicially enriched adjunctions

$$
\begin{tikzcd}
\mathrm{sPre}(\mathrm{Cart}) \arrow[rr, bend left=60, "\Pi_\infty"] \arrow[rr, phantom, bend left=30, "\perp"] \arrow[rr, phantom, "\perp"] \arrow[rr, phantom, bend right=30, "\perp"] \arrow[rr, bend right=25, "\Gamma"'] & & \mathrm{sSet} \arrow[ll, bend right=25, "\mathrm{Disc}"'] \arrow[ll, bend left=60, "\mathrm{CoDisc}"]
\end{tikzcd}
\tag{27}
$$

Proposition 3.8 ([Sch13, Prop 4.1.30 and 4.1.32])**.** Each adjunction in (27) is a simplicial Quillen adjunction, where sPre(Cart) is given the Čech model structure $\check{\mathbb{H}}$, and sSet is given the Kan-Quillen model structure.

In [Sch13], Schreiber defines the three following endofunctors on the category of simplicial presheaves on Cart:

$$
\begin{aligned}
\textstyle\int &= \mathrm{Disc} \circ \Pi_\infty \\
\flat &= \mathrm{Disc} \circ \Gamma \\
\sharp &= \mathrm{CoDisc} \circ \Gamma
\end{aligned}
\tag{28}
$$

called **shape**, **flat** and **sharp** respectively.

Remark 3.9. Using the name shape functor for both Π_∞ and $\int$ is justified by remembering that $\mathrm{Disc} : \mathrm{sSet} \to \check{\mathbb{H}}$ is fully faithful.

They give another pair of simplicial Quillen adjunctions

$$
\begin{tikzcd}
\check{\mathbb{H}} \arrow[rr, "\flat"] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & \check{\mathbb{H}} \arrow[ll, bend right=60, "\int"'] \arrow[ll, bend left=60, "\sharp"]
\end{tikzcd}
\tag{29}
$$

Let us focus further on the shape functor Π_∞. Let Δ^k_a denote the cartesian space defined by

$$
\Delta^k_a = \left\{ (x_0, \ldots, x_k) \in \mathbb{R}^{k+1} \ : \ \sum_{i=0}^{k} x_i = 1 \right\}.
\tag{30}
$$

We call these **affine simplices**.

Let $\mathrm{Sing}_\infty : \mathrm{sPre}(\mathrm{Cart}) \to \mathrm{sSet}$ be the functor defined objectwise by

$$
\mathrm{Sing}_\infty(X) = \mathrm{hocolim}_{\Delta^{op}} \left(\ X(\Delta^0_a) \leftleftarrows X(\Delta^1_a) \ \text{\(\Lleftarrow\)} \ X(\Delta^2_a) \ \cdots \ \right),
\tag{31}
$$

where if we wish to be concrete, we can use the model of the homotopy colimit given by taking the diagonal of the above bisimplicial set. We call this the **smooth singular complex functor**.

Lemma 3.10. The functor Sing_∞ sends objectwise weak equivalences of simplicial presheaves to weak equivalences.

Proof. This follows from [GJ12, Proposition 1.9] and taking the diagonal to model the homotopy colimit. □

Proposition 3.11. There are natural weak equivalences between the functors

$$
\mathrm{Sing}_\infty \simeq \mathrm{Sing}_\infty \circ Q \simeq \Pi_\infty \circ Q
\tag{32}
$$

where Q denotes a cofibrant replacement functor for $\check{\mathbb{H}}$.

Proof. We mirror the proof given in [Bun22, Remark 4.12], and since the left hand weak equivalence is shown there, we only prove the middle weak equivalence. If we restrict the smooth singular complex functor along the Yoneda embedding $\mathsf{Cart} \hookrightarrow \mathsf{sPre}(\mathsf{Cart})$, then we obtain a functor $\mathrm{Sing}_\infty : \mathsf{Cart} \to \mathsf{sSet}$, and there is a natural weak equivalence of functors $\mathrm{Sing}_\infty \xrightarrow{\sim} *$, where $* : \mathsf{Cart} \to \mathsf{sSet}$ is the constant functor on a point $* = \Delta^0$, this follows from [Bun22, Proposition 3.11]. Since all simplicial sets are cofibrant in the Quillen model structure on sSet, [Rie14, Corollary 5.2.5] implies that this induces a natural weak equivalence

$$B(X, \mathsf{Cart}, \mathrm{Sing}_\infty) \xrightarrow{\sim} B(X, \mathsf{Cart}, *) \tag{33}$$

of simplicial presheaves for every $X \in \mathsf{sPre}(\mathsf{Cart})$. But $B(X, \mathsf{Cart}, *) \cong \mathrm{colim}_{\mathsf{Cart}^{op}} B(X, \mathsf{Cart}, y) \cong \mathrm{colim}_{\mathsf{Cart}^{op}} QX \cong \Pi_\infty QX$. This follows from the fact that $B(X, \mathsf{Cart}, y) \cong QX$, which is just repackaging the definition of QX. Since Sing_∞ is a left adjoint, we have

$$B(X, \mathsf{Cart}, \mathrm{Sing}_\infty) \cong \mathrm{Sing}_\infty B(X, \mathsf{Cart}, y) \cong \mathrm{Sing}_\infty QX.$$

This gives the second natural weak equivalence above. □

By Proposition 3.11, we will often refer to $\mathrm{Sing}_\infty(X)$ as the shape of X as well. The shape functor has many wonderful properties. While we will not need all of the following results on the shape functor for this paper, we provide a concise listing of them here as such results are scattered throughout the literature.

Remark 3.12. Since $\int$ is just applying the shape functor and then treating the resulting simplicial set as a constant simplicial presheaf, along with the fact that $\int \simeq \mathrm{Disc} \circ \mathrm{Sing}_\infty$, we will blur the distinction between $\Pi_\infty, \mathrm{Sing}_\infty$ and $\int$. We will use $\int$ when we wish to be ambiguous about which particular model of the shape functor we wish to use.

Theorem 3.13 ([Bun21b, Theorem 4.15])**.** Let M be a finite dimensional smooth manifold, thought of as a simplicial presheaf on Cart, then

$$\int M \simeq \mathrm{Sing}(M^{top}) \tag{34}$$

where M^{top} is the underlying topological space of M and $\mathrm{Sing} : \mathsf{Top} \to \mathsf{sSet}$ is the classical singular complex. In other words, the shape of M is its underlying homotopy type.

Remark 3.14. It should be noted that Theorem 3.13 is really a consequence of a classical result known as the nerve theorem [Bor48].

Proposition 3.15 ([Pav22b, Example 14.1])**.** Let Ω^n_{cl} denote the sheaf on Cart of closed differential n-forms. Its shape is

$$\int \Omega^n_{\mathrm{cl}} \simeq \mathbf{B}^n \mathbb{R}^\delta. \tag{35}$$

Lemma 3.16. Let K be a simplicial set, then the counit

$$\varepsilon_K : (\Pi_\infty \circ \mathrm{Disc})(K) \to K. \tag{36}$$

is an isomorphism. In other words, for discrete simplicial presheaves K_c, we have

$$\int K_c \cong K.$$

Proof. This follows from the fact that Disc is fully faithful, and the unit of an adjunction where the right adjoint is fully faithful is an isomorphism. □

Proposition 3.17 ([Car15, Theorem 3.4])**.** Let M denote a simplicial manifold. Let $\underline{M}$ denote the simplicial presheaf on Cart defined degreewise by

$$\underline{M}(U)_k = C^\infty(U, M_k).$$

Then the shape of $\underline{M}$ is

$$\int \underline{M} \simeq \|M\| \tag{37}$$

where $\|M\|$ denotes the homotopy type of the "fat" geometric realization of M, see [Car15, Section 3.2].

Remark 3.18. The above result is especially interesting when M is the nerve of a Lie groupoid, as this says that the shape of a Lie groupoid (thought of as a simplicial presheaf) is weak equivalent to the homotopy type of the Lie groupoid's classifying space [Car15, Section 2.2].

Remark 3.19. We should also mention that the shape functor has been used to great effect in what is now called the Smooth Oka Principle. See the following references [BBP22], [SS21], [Clo23], [Pav22a, Section 10].

Let $\mathrm{Sing}_D : \mathsf{Diff} \to \mathsf{sSet}$ be the functor defined levelwise by

$$\mathrm{Sing}_D(X)_n = \mathsf{Diff}(\Delta^n_a, X). \tag{38}$$

We call this the **diffeological singular complex**. Note that by using the diagonal as a model for the homotopy colimit, for a diffeological space X (actually any sheaf of sets on Cart), we have

$$\mathrm{Sing}_D(X) \cong \mathrm{Sing}_\infty(X). \tag{39}$$

Proposition 3.20. Let X be a diffeological space, then

$$\Pi_\infty(QX) \simeq \mathrm{Sing}_D(X) \simeq N\mathsf{Plot}(X) \tag{40}$$

where $N\mathsf{Plot}(X)$ is the nerve of the category of plots of X.

Proof. We will only prove the second weak equivalence, as the first holds by the previous discussion. It is shown in [Bun22, Proposition 3.6] that Sing_∞ has a left and right adjoint (though only the right adjoint forms a Quillen adjunction), therefore we have

$$\begin{aligned}
\Pi_\infty(QX) &= \Pi_\infty\left(\int^{n\in\Delta} \coprod_{N\mathsf{Plot}(X)_n} yU_{p_n} \otimes \Delta^n \right) \\
&\cong \int^{n\in\Delta} \coprod_{N\mathsf{Plot}(X)_n} \Pi_\infty(yU_{p_n}) \times (\Pi_\infty \circ \mathrm{Disc})(\Delta^n) \\
&\simeq \int^{n\in\Delta} \coprod_{N\mathsf{Plot}(X)_n} * \times \Delta^n \\
&\cong N\mathsf{Plot}(X).
\end{aligned} \tag{41}$$

where the weak equivalence is given by Theorem 3.13 and Lemma 3.16. □

4. The Irrational Torus

In this section, we will show that if $K \subset \mathbb{R}$ is a diffeologically discrete subgroup of the real numbers, then the infinity stack cohomology of the irrational torus $T_K = \mathbb{R}^n/K$ (for any $n \geq 1$) with values in $\mathbb{R}^\delta$, is isomorphic to the group cohomology of K with values in $\mathbb{R}^\delta$. This was first proved by Iglesias-Zemmour [Igl23, Page 15] with his own version of diffeological Čech cohomology, which we will refer to as PIZ cohomology. In [Min22] we found maps between ∞-stack cohomology and PIZ cohomology, and showed that one of these maps is a retract, but it is still an open question as to whether these two cohomologies are isomorphic.

The motivation for this section is two-fold. One is to support the conjecture that ∞-stack cohomology is isomorphic to PIZ cohomology. We do this by showing that they agree on one of the most important class of examples of diffeological spaces, the irrational tori. The second motivation is to show the power of ∞-topos theory and in particular ∞-stack cohomology through the use of the shape operation. The proof of Theorem 4.4 is completely different than that in [Igl23], and it is conceptually more straightforward.

Definition 4.1. Suppose that $K \subset \mathbb{R}^n$ is a subgroup, and furthermore, when K is given the subset diffeology of $\mathbb{R}$, it coincides with the discrete diffeology, so that every plot is constant. We call its quotient $T_K = \mathbb{R}^n/K$ the n-dimensional K**-irrational torus**[8].

We define the quotient map $\pi : \mathbb{R}^n \to T_K$ of diffeological spaces. This map is a diffeological principal K-bundle [Igl13, Article 8.15], where K is given the discrete diffeology. By the discussion in Example 3.7, the bundle π is classified by a map of ∞-stacks $g_K : T_K \to N\mathsf{DiffPrin}_K$. Furthermore, by [Min22, Corollary 6.12], we obtain the following cube, where the front face and back face are homotopy pullback squares and the maps going from the back face to the front face are all objectwise weak equivalences

(42)
$$\begin{tikzcd}
\widetilde{\mathbb{R}^n} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{E}K \arrow[dr] \arrow[dd] & \\
& \mathbb{R}^n \arrow[rr] \arrow[dd, "\pi"] & & * \arrow[dd] \\
QT_K \arrow[rr] \arrow[dr] & & \mathbf{B}K \arrow[dr] & \\
& T_K \arrow[rr, "g_K"'] & & N\mathsf{DiffPrin}_K
\end{tikzcd}$$

Now if we apply the shape functor to this cube, the back face remains a homotopy pullback by the following result.

Proposition 4.2 ([SS21, Proposition 3.3.8])**.** Let K be a simplicial set, and let $f : X \to \mathrm{Disc}(K)$ and $g : Y \to \mathrm{Disc}(K)$ be maps of simplicial presheaves on Cart. Then there is a Čech weak equivalence

(43)
$$(\textstyle\int X) \times^h_{\mathrm{Disc}(K)} (\textstyle\int Y) \simeq \textstyle\int \left(X \times^h_{\mathrm{Disc}(K)} Y \right)$$

between the homotopy pullbacks of the maps $\int f$ and $\int g$ and the shape of the homotopy pullback of the maps f and g.

Remark 4.3. For a model-category theoretic proof of Proposition 4.2, use the argument of [Sch13, Theorem 4.1.34], and mirror the argument of [Sch13, Proposition 4.1.35] model categorically.

Therefore the front face must also be a homotopy pullback square. Now by Lemma 3.10, Lemma 3.16, and Theorem 3.13 it follows that

(44)
$$\textstyle\int \widetilde{\mathbb{R}^n} \simeq \int \mathbb{R}^n \simeq *, \qquad \int QT_K \simeq \int T_K, \qquad \int \mathbf{E}K \simeq \int * \cong *, \qquad \mathbf{B}K \cong \int \mathbf{B}K \simeq \int \mathsf{DiffPrin}_K.$$

From this we obtain the main result of this section.

Theorem 4.4. There is an isomorphism

(45)
$$\check{H}^n_\infty(T_K, \mathbb{R}^\delta) \cong H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta)$$

of abelian groups, for every $n \geq 0$, where $\mathbb{R}^\delta$ denotes the discrete group of real numbers, and where $H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta)$ denotes the group cohomology of K with coefficients in $\mathbb{R}^\delta$.

Proof. First we note that since $\mathbb{R}^\delta$ is discrete, $\mathbb{R}^\delta \cong \flat \mathbb{R}^\delta$. Similarly, $\mathbf{B}^n\mathbb{R}^\delta \cong \flat \mathbf{B}^n \mathbb{R}^\delta$ for all $n \geq 0$. Thus

(46)
$$\mathbb{R}\check{\mathrm{H}}(T_K, \mathbf{B}^n\mathbb{R}^\delta) \cong \mathbb{R}\check{\mathrm{H}}(T_K, \flat\mathbf{B}^n\mathbb{R}^\delta) \cong \mathbb{R}\check{\mathrm{H}}(\textstyle\int T_K, \mathbf{B}^n\mathbb{R}^\delta).$$

Now from Proposition 3.11, there is a weak equivalence $\mathrm{Sing}_D(T_K) \to \int T_K$ of simplicial sets, and by [CW14, Proposition 4.30], $\mathrm{Sing}_D(T_K)$ is a Kan complex. Now $\int N\mathsf{DiffPrin}_K$ is a Kan

[8] The word irrational comes from the example where $n = 1$, and $K = \mathbb{Z} + \alpha\mathbb{Z}$ with α an irrational number. This is the most studied example of an irrational torus in diffeology. Interestingly, $\mathbb{Z} + \alpha\mathbb{Z} \subset \mathbb{R}$ with the subset topology is dense in $\mathbb{R}$, hence not discrete, however the subset diffeology is discrete.

complex since it is a simplicial abelian group. Furthermore since T_K is diffeologically connected [Igl23, Section II], $\mathrm{Sing}_D(T_K)$ is connected. Thus the composite map $\mathrm{Sing}_D(T_K) \to \int N\mathsf{DiffPrin}_K$ is a map of connected Kan complexes whose homotopy fiber is contractible. Therefore by the long exact sequence of homotopy groups [Cis19, Theorem 3.8.12], it is a weak equivalence, which implies that $\int g_K$ is a weak equivalence. Therefore we have

$$\mathbb{R}\check{\mathbb{H}}(\textstyle\int T_K, \mathbf{B}^n\mathbb{R}^\delta) \simeq \mathbb{R}\check{\mathbb{H}}(\int N\mathsf{DiffPrin}_K, \mathbf{B}^n\mathbb{R}^\delta) \simeq \mathbb{R}\check{\mathbb{H}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta). \tag{47}$$

Which implies that

$$\check{H}^n_\infty(T_K, \mathbb{R}^\delta) = \pi_0\mathbb{R}\check{\mathbb{H}}(\textstyle\int T_K, \mathbf{B}^n\mathbb{R}^\delta) \cong \pi_0\mathbb{R}\check{\mathbb{H}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta) = \check{H}^n_\infty(\mathbf{B}K, \mathbb{R}^\delta). \tag{48}$$

However since $\mathbf{B}K$ and $\mathbf{B}^n\mathbb{R}^\delta$ are discrete, Disc is fully faithful and $\mathbf{B}^n\mathbb{R}^\delta$ is a Kan complex, we have

$$\mathbb{R}\check{\mathbb{H}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta) \simeq \underline{\mathsf{sSet}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta),$$

where $\underline{\mathsf{sSet}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta)$ is the usual simplicial set function complex. It is then well known (see [Wei95, Example 8.2.3] for instance) that

$$\pi_0\underline{\mathsf{sSet}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta) \cong H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta). \tag{49}$$

This proves the theorem. □

5. The Dold-Kan Correspondence

In this section, we discuss the Dold-Kan correspondence, which is central to Section 7.

Remark 5.1. For the remainder of this paper, by a vector space, we mean a real vector space, not necessarily of finite dimension. By a chain complex we mean a non-negatively graded chain complex of vector spaces. Let Ch denote the category of chain complexes.

Definition 5.2. Let Vect denote the category whose objects are vector spaces and whose morphisms are linear maps. Let $\mathsf{sVect} := \mathsf{Vect}^{\Delta^{op}}$ denote the category of simplicial vector spaces.

Proposition 5.3 ([GS06, Proposition 4.2 and Theorem 4.13], [Jar03, Section 1])**.** The category sVect admits a proper, combinatorial, simplicial model category structure where a morphism $f : X \to Y$ is a

(1) weak equivalence if it is a weak homotopy equivalence of the underlying simplicial sets,
(2) fibration if it is a Kan fibration of the underlying simplicial sets,
(3) cofibration if it is degreewise a monomorphsim.

We call this the **Kan-Quillen** model structure[9] on sVect.

Proposition 5.4 ([GS06, Theorem 1.5], [Jar03, Section 1])**.** The category Ch admits a proper, combinatorial, simplicial model category structure where a morphism $f : C \to D$ is a

(1) weak equivalence if it is a quasi-isomorphism of chain complexes, and
(2) fibration if $f_k : C_k \to D_k$ is surjective in degrees $k \geq 1$.
(3) cofibration if it is degreewise a monomorphism[10].

We call this the **projective model structure**[11] on chain complexes.

There is an adjoint pair of functors,

$$\begin{tikzcd} \mathsf{Ch} \arrow[r, bend left, "\mathsf{DK}"] \arrow[r, phantom, "\perp"] & \mathsf{sVect} \arrow[l, bend left, "N"] \end{tikzcd} \tag{50}$$

[9]Note that every object in sVect is fibrant and cofibrant.

[10]For the projective model structure on chain complexes of R-modules for a general commutative ring R, we require these to be degreewise monomorphisms with projective cokernel. Since all vector spaces are projective as $\mathbb{R}$-modules, this condition is always satisfied.

[11]Note that every object in Ch is fibrant and cofibrant.

which by the Dold-Kan correspondence [GS06, Theorem 4.1] form an adjoint equivalence[12].

Lemma 5.5 ([SS03, Section 4.1])**.** The adjunction $N \dashv \mathsf{DK}$ is a Quillen adjunction ([Hir09, Definition 8.5.2]) between the model category structures on Ch and sVect. Furthermore, the functors form a Quillen equivalence ([Hir09, Definition 8.5.20]). In fact, since N and DK form an adjoint equivalence, it follows that $\mathsf{DK} \dashv N$ is also an adjoint equivalence. Furthermore N and DK are both left and right Quillen functors.

If we consider the category of simplicial sets sSet with its usual Kan-Quillen model structure [GS06, Theorem 1.22], there is a simplicial Quillen adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "{\mathbb{R}[-]}"] \arrow[r, phantom, "\perp"] & \mathsf{sVect} \arrow[l, bend left, "U"] \end{tikzcd} \tag{51}$$

where U denotes the forgetful functor, and $\mathbb{R}[-]$ denotes the functor that sends a simplicial set X to the free simplicial vector space $\mathbb{R}X$, defined degreewise by $\mathbb{R}X_n = \mathbb{R}(X_n)$, where $\mathbb{R}(X_n)$ is the free vector space on the set X_n. Thus we obtain a Quillen adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "{N\mathbb{R}[-]}"] \arrow[r, phantom, "\perp"] & \mathsf{Ch} \arrow[l, bend left, "U\mathsf{DK}"] \end{tikzcd} \tag{52}$$

which is furthermore a simplicial Quillen adjunction[13].

Note that the Dold-Kan correspondence also provides a simplicial enrichment of Ch. Indeed, suppose C and D are chain complexes, then let $\underline{\mathsf{Ch}}(C,D)$ denote the simplicial vector space defined degreewise by

$$\underline{\mathsf{Ch}}(C,D)_k = \mathsf{Ch}(N\mathbb{R}\Delta^k \otimes C, D). \tag{53}$$

This makes the Dold-Kan correspondence an enriched adjoint equivalence. This is also the simplicial enrichment mentioned in Proposition 5.4.

This also supplies Ch with tensoring and cotensoring over sSet. Namely if K is a simplicial set and C is a chain complex then $C \otimes K$ is the chain complex $C \otimes N\mathbb{R}K$, and $C^K = N\mathbb{R}\underline{\mathsf{Ch}}(N\mathbb{R}K, C)$.

Now the category of chain complexes Ch is also enriched over itself. Indeed, if C and D are chain complexes, then let $\underline{\mathsf{Map}_{\mathsf{Ch}}}(C,D)$ denote the chain complex defined as follows. First let us define the unbounded chain complex $\underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$ defined in degree $k \in \mathbb{Z}$ by

$$\underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)_k = \prod_{i \geq 0} \mathsf{Vect}(C_i, D_{i+k}), \tag{54}$$

with $d : \underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)_k \to \underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)_{k-1}$ defined for a map f by

$$df = d_D f - (-1)^k f d_C.$$

We call an element of degree k in $\underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$ a degree k map from C to D.

Definition 5.6. If C is an unbounded ($\mathbb{Z}$-graded) chain complex, then let $\tau_{\geq 0}C$ denote the chain complex defined degreewise by $(\tau_{\geq 0}C)_k = C_k$ for $k > 0$, and $(\tau_{\geq 0}C)_0 = Z_0 C$, the set of 0-cycles of C, i.e. those $x \in C_0$ such that $dx = 0$, the differential on $\tau_{\geq 0}C$ is induced by the differential on C. We call $\tau_{\geq 0}C$ the **smart truncation** of C.

Now given chain complexes C and D, let

$$\underline{\mathsf{Map}_{\mathsf{Ch}}}(C,D) = \tau_{\geq 0}\underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D), \tag{55}$$

[12]This result actually holds for chain complexes and simplicial objects taking values in any idempotent complete, additive category, see [Lur17, Theorem 1.2.3.7].

[13]We will often omit the functor U in our notation.

denote the smart truncation applied to $\underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$. This means that $\underline{\mathsf{Map}_{\mathsf{Ch}}}(C,D)_k = \underline{\mathsf{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$ for $k > 0$, and $\underline{\mathsf{Map}_{\mathsf{Ch}}}(C,D)_0 \cong \mathsf{Ch}(C,D)$. We refer to $\underline{\mathsf{Map}_{\mathsf{Ch}}}(C,D)$ as the **mapping chain complex** between C and D.

Lemma 5.7 ([Opa21, Example 4.3.2])**.** Let C and D be chain complexes, then we have an isomorphism of simplicial vector spaces

$$\mathsf{DK}\,\underline{\mathsf{Map}_{\mathsf{Ch}}}(C,D) \cong \underline{\mathsf{Ch}}(C,D). \tag{56}$$

Further, this provides an isomorphism

$$\underline{\mathsf{Map}_{\mathsf{Ch}}}(C,D) \cong N\underline{\mathsf{Ch}}(C,D). \tag{57}$$

An explicit description for the path space of a chain complex C, equivalently the cotensoring C^{Δ^1}, is given in Appendix C.

Definition 5.8. Let $\mathcal{C}$ be a small category. Then let $\mathsf{ChPre}(\mathcal{C})$ denote the category whose objects are functors $\mathcal{C}^{op} \to \mathsf{Ch}$, and whose morphisms are natural transformations. We call such functors presheaves of chain complexes.

Proposition 5.9 ([Hir09, Section 11.6])**.** The category $\mathsf{ChPre}(\mathcal{C})$ admits a proper, combinatorial, simplicial model category structure where a morphism $f : C \to D$ is a

(1) weak equivalence if it objectwise a weak equivalence in the projective model structure on chain complexes, and
(2) fibration if it is objectwise a fibration in the projective model structure on chain complexes.

We refer to this as the **(global) projective model structure** on presheaves of chain complexes.

Thus we obtain a similar simplicial Quillen pair

$$\begin{tikzcd} \mathsf{sPre}(\mathcal{C}) \arrow[r, bend left, "N\mathbb{R}[-]"] \arrow[r, phantom, "\perp"] & \mathsf{ChPre}(\mathcal{C}) \arrow[l, bend left, "U\mathsf{DK}"] \end{tikzcd} \tag{58}$$

where $\mathsf{sPre}(\mathcal{C})$ is equipped with the projective model structure. In Appendix C, we will use (58), along with Proposition 3.5, to compute homotopy pullbacks in $\check{\mathbb{H}}$.

6. Examples of ∞-stacks

In this section we detail the ∞-stacks involved in this paper, and examine the ∞-stack cohomology of a diffeological space with coefficients in some of these example ∞-stacks.

Example 6.1. Given a finite dimensional smooth manifold M, the functor $U \mapsto C^\infty(U,M)$ defines a sheaf on Cart, and therefore an ∞-stack. The same goes for diffeological spaces. Given a diffeological space X, let X^δ denote the diffeological space with the same underlying set, but equipped with the discrete diffeology. As sheaves we have $X^\delta \cong \mathrm{Disc}(X(*)) = \flat X$.

Example 6.2. The presheaf of differential k-forms Ω^k and the presheaf of closed differential k-forms Ω^k_{cl} are sheaves of vector spaces on Cart for every $k \geq 0$. Thus they are ∞-stacks. The de Rham differential defines a map of ∞-stacks $d : \Omega^k \to \Omega^{k+1}$ for all $k \geq 0$.

There is a canonical map

$$\mathrm{mc}(\mathbb{R}) : \mathbb{R} \to \Omega^1 \tag{59}$$

of ∞-stacks, defined by the Yoneda Lemma as follows. Notice that $\mathbb{R} \in \mathsf{Cart}$, so by the Yoneda lemma, a map $\omega : y\mathbb{R} \to \Omega^1$ is equivalent to an element $\omega \in \Omega^1(\mathbb{R})$. There is a canonical element of the set of 1-forms on $\mathbb{R}$, called the **Maurer-Cartan form** of $\mathbb{R}$. For a general Lie group G, we let $\mathrm{mc}(G)$ denote its Maurer-Cartan form. If we label the coordinate of $\mathbb{R}$ by t, then the Maurer-Cartan form is simply given by $\mathrm{mc}(\mathbb{R}) = dt$. Thus for a cartesian space U, the function $\mathrm{mc}(\mathbb{R})(U) : \mathbb{R}(U) \to \Omega^1(U)$ acts by taking a smooth map $f : U \to \mathbb{R}$ and pulling back the Maurer-Cartan form $f^*\mathrm{mc}(\mathbb{R}) \in \Omega^1(U)$. Note that this is the same thing as df. In other words as maps of ∞-stacks, we have $\mathrm{mc}(\mathbb{R}) = d$.

Example 6.3. Let G be a diffeological group. As discussed in Example 3.7 the presheaf of groupoids $\mathbf{B}G$ given by

$$U \mapsto [C^\infty(U,G) \rightrightarrows *]$$

is a stack, and is objectwise weak equivalent to the stack of diffeological principal G-bundles. We abuse notation as in [Min22, Example 5.13] and also let $\mathbf{B}G$ denote the corresponding simplicial presheaf, which is an ∞-stack. Given a diffeological space X, a G-cocycle on X as in Definition 2.6 is precisely a map $QX \to \mathbf{B}G$ of simplicial presheaves. Theorem 2.7 shows that the resulting groupoid of G-cocycles on X is equivalent to the groupoid of diffeological principal G-bundles on X. Thus $\check{H}^1_\infty(X,G) := \check{H}^0_\infty(X,\mathbf{B}G)$ is the set of isomorphism classes of G-cocycles, which is isomorphic to the set of isomorphism classes of diffeological principal G-bundles.

Example 6.4. If G is a Lie group with Lie algebra $\mathfrak{g}$, then let $\Omega^1(-,\mathfrak{g})//G$ denote the presheaf of groupoids

$$U \mapsto [\Omega^1(U,\mathfrak{g}) \times C^\infty(U,G) \underset{s}{\overset{t}{\rightrightarrows}} \Omega^1(U,\mathfrak{g})]$$

where $t(\omega,g) = \omega$ and $s(\omega,g) = \mathrm{Ad}_g^{-1}(\omega) + g^*\mathrm{mc}(G)$. The nerve of this presheaf of groupoids is an ∞-stack [FSS+12, Proposition 3.2.5][14]. This is the ∞-stack that classifies principal G-bundles with connection. We will often abuse notation and write $\Omega^1(-,\mathfrak{g})//G$ to refer to the presheaf of groupoids and the simplicial presheaf obtained by taking the nerve construction. Note that a map $QX \to \Omega^1(-,\mathfrak{g})//G$ is equivalent to the data of a G-cocycle $g_{f_0} : U_{p_1} \to G$ and a collection $\{A_{p_0}\}_{p_0 \in \mathsf{Plot}(X)}$ of 1-forms $A_{p_0} \in \Omega^1(U_{p_0},\mathfrak{g})$ such that for every map $f_0 : U_{p_1} \to U_{p_0}$ of plots we have

$$A_{p_1} = \mathrm{Ad}_{g_{f_0}}^{-1}(f_0^* A_{p_0}) + g_{f_0}^*\mathrm{mc}(G). \tag{60}$$

Let us call this collection of data $(g,A) = (\{g_{f_0}\},\{A_{p_0}\})$ a **G-cocycle with connection**. We show that this definition of connection is equivalent to the one given in [Wal12, Definition 3.2.1] in Appendix A.

Remark 6.5. The following examples of simplicial presheaves can be checked to be ∞-stacks by using [Pav22a, Corollary 6.2]. One simply needs to notice that the examples that follow are presheaves of bounded chain complexes, and can thus be thought of equivalently as presheaves of cochain complexes, and that the homotopy descent condition for presheaves of cochain complexes is equivalent to the condition of Dold-Kan applied to the presheaves of chain complexes to be ∞-stacks.

Example 6.6. Given a sheaf A of abelian groups on Cart, with $k \geq 1$, the simplicial presheaf $\mathbf{B}^k A$ is obtained by taking Dold-Kan of the presheaf of chain complexes $[A \to 0 \to \cdots \to 0]$. When A is an abelian diffeological group, then $\mathbf{B}A$ is the ∞-stack that classifies diffeological principal A-bundles, as shown in [Min22].

In this paper we will consider the example $\mathbf{B}^k\mathbb{R}$. Given a diffeological space X, a map $QX \to \mathbf{B}^k\mathbb{R}$ consists of a $g \in \mathbb{R}(QX_k)$ such that $\delta g = 0$, see Appendix B. We call these **$\mathbb{R}$-bundle $(k-1)$-gerbes**. Thus a diffeological principal $\mathbb{R}$-bundle is precisely a $\mathbb{R}$-bundle 0-gerbe.

There is a vast literature on bundle gerbes in differential geometry such as [Mur96], [Bun21a], [Ste04]. Typically bundle gerbes are defined as geometric objects, and then shown to define cohomology classes through cocycles such as above. However, giving descriptions of bundle k-gerbes as geometric objects becomes difficult and tedious as k grows. Their description as cocycles is much more economical, and is all we need for this paper. There should be no difficulty in translating between the geometric description of diffeological bundle 1-gerbes, such as in [Wal12] and the cocycle description we give here, but we leave this to future work.

[14] Notice that the definition above is precisely the opposite of the corresponding ∞-stack considered in [FSS+12, Section 3]. This is because of the convention we use in [Min22, Example 5.13]. However this makes no difference on the theory as we show in Appendix A.

Example 6.7. For $k \geq 1$, let $\mathbf{B}^k_\nabla \mathbb{R}$[15] denote the simplicial presheaf obtained by applying Dold-Kan to the following presheaf of chain complexes[16]

$$[\mathbb{R} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^k]. \tag{61}$$

This complex is often referred to as the **Deligne complex** when $\mathbb{R}$ is replaced with $U(1)$ and $d : \mathbb{R} \to \Omega^1$ is replaced with $d\log : U(1) \to \Omega^1$. For $k \geq 2$, $\mathbf{B}^k_\nabla \mathbb{R}$ classifies $\mathbb{R}$-bundle $(k-1)$-gerbes with connection, and ∞-stack cohomology with values in $\mathbf{B}^k_\nabla \mathbb{R}$ is called **pure differential cohomology**[17] in [Jaz21, Definition 3.2.10]. Thus we call $\mathbf{B}^k_\nabla \mathbb{R}$ the **pure k-Deligne complex**.

Given a diffeological space X, we will denote ∞-stack cohomology with values in the pure k-Deligne complex by

$$\check{H}^k_{\infty,\nabla}(X, \mathbb{R}) := \check{H}^0_\infty(X, \mathbf{B}^k_\nabla \mathbb{R}).$$

Let us also note that when $k = 1$, we have an objectwise weak equivalence of ∞-stacks

$$\mathbf{B}_\nabla \mathbb{R} \simeq (\Omega^1(-)//\mathbb{R})^{op}.$$

To see this, note that both of the simplicial presheaves are identical in simplicial degrees 0 and 1. This is because $\mathbb{R}$ is abelian, so $t(\omega, g) = \mathrm{Ad}_g^{-1}(\omega) + g^*\mathrm{mc}(\mathbb{R}) = \omega + dg$ in $(\Omega^1(-)//\mathbb{R})^{op}$, which is precisely the face map $d_0 : \mathbf{B}_\nabla \mathbb{R}_1 \to \mathbf{B}_\nabla \mathbb{R}_0$. Since $(\Omega^1(-)//\mathbb{R})^{op}$ is the nerve of a presheaf of groupoids, it is 2-coskeletal, and therefore its k-homotopy groups are trivial for $k \geq 2$. The objectwise homotopy groups of $\mathbf{B}_\nabla \mathbb{R}$ are given by the objectwise homology of the chain complex by the Dold-Kan correspondence, and thus are also trivial for $k \geq 2$, thus they are objectwise weak equivalent. The distinction between $\Omega^1(-)//\mathbb{R}$ and $(\Omega^1(-)//\mathbb{R})^{op}$ is because of [Min22, Example 5.13], so technically $\mathbf{B}_\nabla \mathbb{R}$ classifies diffeological principal $\mathbb{R}^{op}$-bundles with opposite connection, but this distinction is immaterial to the theory and we sweep it under the rug, and we say that the above classifies diffeological principal $\mathbb{R}$-bundles with connection.

If X is a diffeological space, then as we will see in Example 6.13, a $\mathbb{R}$-bundle $(k-1)$-gerbe with connection on X is given by the data

$$(\omega^k, \omega^{k-1}, \dots, \omega^1, g) \in \Omega^k(QX_0) \oplus \Omega^{k-1}(QX_1) \oplus \cdots \oplus \Omega^1(QX_{k-1}) \oplus \mathbb{R}(QX_k), \tag{62}$$

such $D(\omega^k, \dots, g) = 0$ in the double complex $\Omega^i(QX_j)$, see Appendix B. We will let $[\omega^k, \dots, g]$ denote the isomorphism class it represents in $\check{H}^k_{\infty,\nabla}(X, \mathbb{R})$.

Example 6.8. For $k \geq 1$, consider the ∞-stack $\mathbf{B}^k \mathbb{R}^\delta$. This ∞-stack classifies diffeological principal $\mathbb{R}^\delta$-bundles. However, note that there is a map of presheaves of chain complexes

$$[\mathbb{R}^\delta \to 0 \to \cdots \to 0] \to [\mathbb{R} \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k_{\mathrm{cl}}] \tag{63}$$

given by the inclusion $\mathbb{R}^\delta \hookrightarrow \mathbb{R}$. Furthermore, this map is an objectwise quasi-isomorphism, by the Poincare lemma. Thus we will take the right hand side of (63) to be the model of $\mathbf{B}^k \mathbb{R}^\delta$ we will use for the rest of this paper. From this presentation, it is easy to see that $\mathbf{B}^k \mathbb{R}^\delta$ is equivalently the ∞-stack that classifies diffeological principal $\mathbb{R}$-bundle $(k-1)$-gerbes with flat connection.

Example 6.9. For $k \geq 1$, consider the ∞-stack $\mathbf{B}^k \Omega^1_{\mathrm{cl}}$. There is a map of presheaves of chain complexes

$$[\Omega^1_{\mathrm{cl}} \to 0 \to \cdots \to 0] \to [\Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \xrightarrow{d} \Omega^k_{\mathrm{cl}}] \tag{64}$$

and this map is an objectwise quasi-isomorphism again by the Poincare lemma. We take the right hand side to be the model we will use for $\mathbf{B}^k \Omega^1_{\mathrm{cl}}$ for the rest of the paper.

[15] Note that the $\mathbf{B}^k$ in $\mathbf{B}^k_\nabla \mathbb{R}$ is just notation, it is not actually the delooping of anything.

[16] Note that the way this chain complex is written, Ω^k is in degree 0.

[17] We also recommend [Jaz21, Section 3.2] for a discussion of how pure differential cohomology fits into the hexagon diagram of differential cohomology.

Example 6.10. For $k \geq 1$, let $\Omega^{1\leq \bullet \leq k}$ denote the simplicial presheaf obtained by applying Dold-Kan to the following presheaf of chain complexes

$$[\Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^k]. \tag{65}$$

If X is a diffeological space and A is a k-deloopable ∞-stack, recall that its ∞-stack cohomology is given by

$$\check{H}^k_\infty(X, A) = \mathbb{R}\check{\mathbb{H}}(X, \mathbf{B}^k A).$$

Let us compute an example of ∞-stack cohomology for a diffeological space X with values in the ∞-stack $\mathbf{B}\Omega^1$, as it will be emblematic of how we compute ∞-stack cohomology for all of the relevant examples presented in this section. In Appendix B we go into detail on how to compute such examples.

Example 6.11. Let $A' = \mathbf{B}\Omega^1 = [\Omega^1 \to 0]$ and $A = \mathsf{DK}\, A'$. Let us compute $H^0(X, A)$ for a diffeological space X using Proposition B.3. Consider the double complex $A'(QX)$,

$$\begin{tikzcd}
\Omega^1(QX_0) \arrow[r, "\delta"] \arrow[d] & \Omega^1(QX_1) \arrow[r, "-\delta"] \arrow[d] & \Omega^1(QX_2) \arrow[r] \arrow[d] & \ldots \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd} \tag{66}$$

so that $\mathrm{tot}\, A'(QX)$ is the chain complex

$$\mathrm{tot}\, A'(QX) = [\Omega^1(QX_0) \xrightarrow{D} \ker(-\delta : \Omega^1(QX_1) \to \Omega^1(QX_2))] \tag{67}$$

Thus a 0-cycle in $\mathrm{tot}\, A'(QX)$ consists of a collection $\{\omega_{f_0} \in \Omega^1(U_{p_1})\}_{f_0 : U_{p_1} \to U_{p_0}}$ of 1-forms for every map of plots of X such that $-\delta\omega = 0$, which is equivalent to the condition that for every pair of composable plot maps $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$ we have

$$(\delta\omega)_{(f_1, f_0)} = f_1^* \omega_{f_0} - \omega_{f_0 f_1} + \omega_{f_1} = 0.$$

Two such 0-cocycles are cohomologous $\omega \sim \omega'$ if there exists a collection $\{\lambda_{p_0} \in \Omega^1(U_{p_0})\}_{p_0 \in \mathsf{Plot}(X)}$ of 1-forms for every plot of X such that

$$(D\lambda)_{f_0} = (\delta\lambda)_{f_0} = \omega'_{f_0} - \omega_{f_0}$$

for every map $f_0 : U_{p_1} \to U_{p_0}$ of plots of X. Thus $\check{H}^0_\infty(X, \mathbf{B}\Omega^1) = \check{H}^1_\infty(X, \Omega^1)$ is precisely analogous to the term $H^{1,1}_\delta$ in [Igl23, Section 4.4].

Example 6.12. Let us compute $\check{H}^1_{\infty,\nabla}(X, \mathbb{R}) := \check{H}^0_\infty(X, \mathbf{B}_\nabla \mathbb{R})$ for a diffeological space X. Consider the double complex

$$\begin{tikzcd}
\mathbb{R}(QX_0) \arrow[r, "\delta"] \arrow[d, "d"'] & \mathbb{R}(QX_1) \arrow[r, "-\delta"] \arrow[d, "d"'] & \mathbb{R}(QX_2) \arrow[r] \arrow[d, "d"'] & \ldots \\
\Omega^1(QX_0) \arrow[r, "-\delta"] & \Omega^1(QX_1) \arrow[r, "\delta"] & \Omega^1(QX_2) \arrow[r] & \ldots
\end{tikzcd} \tag{68}$$

where $\mathbb{R}(QX_i) = C^\infty(QX_i, \mathbb{R})$. A 0-cocycle is the data of a map $g : QX_1 \to \mathbb{R}$ and a 1-form $A \in \Omega^1(QX_0)$ such that $-\delta g = 0$ and $-\delta A = dg$. The condition $-\delta g = \delta g = 0$ is equivalent to the condition $g_{f_0 f_1} = f_1^* g_{f_0} + g_{f_1}$ for every pair of composable plot maps $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$, which is precisely the condition for g to be a cocycle defining a diffeological principal $\mathbb{R}$-bundle on X, see [Min22, Section 5]. The condition $\delta A = dg$ is equivalent to the condition that for every map of plots $f_0 : U_{p_1} \to U_{p_0}$ we have $A_{p_1} - f_0^* A_{p_0} = dg_{f_0}$, which is precisely the equation for a connection on a diffeological principal $\mathbb{R}$-bundle, see Appendix A. Given two 0-cocycles (A, g) and (A', g'), a 1-coboundary consists of an element $h \in \mathbb{R}(QX_0)$ such that $\delta h = g' - g$ and $dh = A' - A$. This is precisely the definition of a morphism of G-cocycles with connection, see Definition A.1. Thus an element of $\check{H}^1_{\infty,\nabla}(X, \mathbb{R})$ is an isomorphism class of a diffeological principal $\mathbb{R}$-bundle on X with connection.

Example 6.13. Let us compute $\check{H}^2_{\infty,\nabla}(X,\mathbb{R})$ for a diffeological space X. Consider the double complex

$$\begin{tikzcd}
\mathbb{R}(QX_0) \arrow[r, "-\delta"] \arrow[d, "d"'] & \mathbb{R}(QX_1) \arrow[r, "\delta"] \arrow[d, "d"'] & \mathbb{R}(QX_2) \arrow[r] \arrow[d, "d"'] & \dots \\
\Omega^1(QX_0) \arrow[r, "\delta"] \arrow[d, "d"'] & \Omega^1(QX_1) \arrow[r, "-\delta"] \arrow[d, "d"'] & \Omega^1(QX_2) \arrow[r] \arrow[d, "d"'] & \dots \\
\Omega^2(QX_0) \arrow[r, "-\delta"] & \Omega^2(QX_1) \arrow[r, "\delta"] & \Omega^2(QX_2) \arrow[r] & \dots
\end{tikzcd} \tag{69}$$

Then a 0-cycle in $\operatorname{tot}\mathbf{B}^2_\nabla\mathbb{R}(QX)$ is an element $(\omega,A,g)\in\Omega^2(QX_0)\oplus\Omega^1(QX_1)\oplus\mathbb{R}(QX_2)$ such that $D(\omega,A,g)=0$. This is equivalent to the equations $-\delta\omega=dA$, $-\delta A=dg$ and $-\delta g=0$. The equation $-\delta g=\delta g=0$ is equivalent to the condition that for every triple of composable maps of plots (f_2,f_1,f_0) of X, we have

$$(\delta g)_{(f_2,f_1,f_0)} = f_2^* g_{(f_1,f_0)} - g_{(f_1 f_2,f_0)} + g_{(f_2,f_0 f_1)} - g_{(f_2,f_1)} = 0. \tag{70}$$

This is the diffeological analogue of the cocycle data of a $\mathbb{R}$-bundle gerbe on X. The other two equations $-\delta\omega=dA$ and $-\delta A=dg$ are the diffeological analogue of the cocycle data of a connection on a $\mathbb{R}$-bundle gerbe. Thus $\check{H}^2_{\infty,\nabla}(X,\mathbb{R})=\check{H}^0_\infty(X,\mathbf{B}^2_\nabla\mathbb{R})$ can be taken as the definition of the abelian group of isomorphism classes of diffeological $\mathbb{R}$-bundle 1-gerbes with connection on X. The story for $k\geq 2$ is exactly analogous, and thus we take our definition of the abelian group of isomorphism classes of diffeological $\mathbb{R}$-bundle $(k-1)$-gerbes with connection to be $\check{H}^k_{\infty,\nabla}(X,\mathbb{R})=\check{H}^0_\infty(X,\mathbf{B}^k_\nabla\mathbb{R})$.

7. The Čech de Rham Obstruction

In this section, we obtain a diffeological Čech-de Rham obstruction exact sequence in every degree from a homotopy pullback diagram of ∞-stacks. In degree 1, our exact sequence is analogous to [Igl23].

Let X be a diffeological space. In [Igl23], Iglesias-Zemmour constructs the following exact sequence of vector spaces

$$0\to H^1_{dR}(X)\to \check{H}^1_{PIZ}(X,\mathbb{R}^\delta)\to {}^dE_2^{1,0}(X)\xrightarrow{c_1} H^2_{dR}(X)\to \check{H}^2_{PIZ}(X,\mathbb{R}^\delta) \tag{71}$$

using the five term exact sequence coming from a diffeological version of the Čech-de Rham bicomplex spectral sequence. The vector space ${}^dE_2^{1,0}(X)$ is the subspace of isomorphism classes of diffeological principal $\mathbb{R}$-bundles on X that admit a connection, and the vector spaces $\check{H}^k_{\mathrm{PIZ}}(X,\mathbb{R}^\delta)$ are Iglesias-Zemmour's version of diffeological Čech cohomology, which we refer to as PIZ cohomology. The relationship between PIZ cohomology and ∞-stack cohomology is only partially understood, see [Min22, Section 5].

The exact sequence (71) demonstrates the obstruction to the Čech-de Rham Theorem holding for diffeological spaces. For finite dimensional smooth manifolds, all principal $\mathbb{R}$-bundles are trivial, as they have contractible fiber, and thus the obstruction vanishes. However, there are diffeological spaces (the irrational torus for example) that have nontrivial principal $\mathbb{R}$-bundles that admit connections [Igl13, Article 8.39].

We construct and geometrically interpret the obstruction to the Čech-de Rham isomorphism in each degree $k\geq 1$ via ∞-stacks.

Theorem 7.1. For every $k \geq 1$, there exists a commutative diagram of ∞-stacks of the following form

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathbf{B}^k\mathbb{R}^\delta \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \mathbf{B}^k_\nabla\mathbb{R} \arrow[r] \arrow[d] & \Omega^{k+1}_{\mathrm{cl}} \arrow[r] \arrow[d] & \Omega^{k+1} \arrow[d] \\
 & \mathbf{B}^k\mathbb{R} \arrow[r] \arrow[d] & \mathbf{B}^k\Omega^1_{\mathrm{cl}} \arrow[r] \arrow[d] & \Omega^{1\leq\bullet\leq k+1} \arrow[d] \\
 & * \arrow[r] & \mathbf{B}^{k+1}\mathbb{R}^\delta \arrow[r] & \mathbf{B}^{k+1}_\nabla\mathbb{R}
\end{tikzcd} \tag{72}$$

furthermore every commutative square in this diagram is a homotopy pullback square in $\check{\mathbb{H}}$.

We prove Theorem 7.1 in Appendix C.

Corollary 7.2. For every diffeological space X, there is an exact sequence of vector spaces[18]

$$0 \to \check{H}^k_\infty(X,\mathbb{R}^\delta) \to \check{H}^k_{\infty,\nabla}(X,\mathbb{R}) \to \Omega^{k+1}_{\mathrm{cl}}(X) \to \check{H}^{k+1}_\infty(X,\mathbb{R}^\delta). \tag{73}$$

Proof. This follows from Theorem 7.1 and Lemma C.6. □

Let us explore the consequences of Corollary 7.2 in the case where X is the irrational torus. Let $K = \mathbb{Z} + \alpha\mathbb{Z}$ be the subgroup of $\mathbb{R}$ consisting of those $x \in \mathbb{R}$ of the form $n + \alpha m$ where n, m are integers and α is an irrational number. Let $T_\alpha = \mathbb{R}/\mathbb{Z} + \alpha\mathbb{Z}$. We can fully compute the de Rham and ∞-stack cohomology of T_α. Every differential form on T_α is closed [Igl13, Exercise 119] so $\Omega^k(T_\alpha) = \Omega^k_{\mathrm{cl}}(T_\alpha) = H^k_{\mathrm{dR}}(T_\alpha)$, and furthermore $\Omega^k_{\mathrm{cl}}(T_\alpha) \cong \Lambda^k\mathbb{R}$ by [Igl13, Exercise 105]. Therefore we have

$$\Omega^k_{\mathrm{cl}}(T_\alpha) = H^k_{\mathrm{dR}}(T_\alpha) \cong \begin{cases} \mathbb{R}, & k = 0, 1 \\ 0, & k > 1. \end{cases} \tag{74}$$

Now by Theorem 4.4, we have

$$\check{H}^k_\infty(T_\alpha,\mathbb{R}^\delta) \cong \check{H}^0_\infty(\mathbf{B}(\mathbb{Z}+\alpha\mathbb{Z}),\mathbf{B}^k\mathbb{R}^\delta) \cong \check{H}^0_\infty(\mathbf{B}\mathbb{Z}^2,\mathbf{B}^k\mathbb{R}^\delta) \cong \check{H}^k_\infty(T^2,\mathbb{R}^\delta) \cong \begin{cases} \mathbb{R}, & k = 0, 2 \\ \mathbb{R}^2, & k = 1 \\ 0, & k > 2, \end{cases} \tag{75}$$

where T^2 denotes the usual 2-dimensional torus.

From Corollary 7.2, setting $k = 1$, we obtain the exact sequence

$$0 \to \check{H}^1_\infty(T_\alpha,\mathbb{R}^\delta) \to \check{H}^1_{\infty,\nabla}(T_\alpha,\mathbb{R}) \to \Omega^2_{\mathrm{cl}}(T_\alpha) \to \check{H}^2_\infty(T_\alpha,\mathbb{R}^\delta). \tag{76}$$

Since $\Omega^2_{\mathrm{cl}}(T_\alpha) = 0$, this implies that $\check{H}^1_{\infty,\nabla}(T_\alpha,\mathbb{R}) \cong \check{H}^1_\infty(T_\alpha,\mathbb{R}^\delta) \cong \mathbb{R}^2$.

From Corollary 7.2, setting $k = 2$, we obtain the exact sequence

$$0 \to \check{H}^2_\infty(T_\alpha,\mathbb{R}^\delta) \to \check{H}^2_{\infty,\nabla}(T_\alpha,\mathbb{R}) \to \Omega^3_{\mathrm{cl}}(T_\alpha) \to \check{H}^3_\infty(T_\alpha,\mathbb{R}^\delta), \tag{77}$$

but $\Omega^3_{\mathrm{cl}}(T_\alpha) \cong H^3_{\mathrm{dR}}(T_\alpha) \cong 0 \cong \check{H}^3_\infty(T_\alpha,\mathbb{R}^\delta)$, thus $\check{H}^2_{\infty,\nabla}(T_\alpha,\mathbb{R}) \cong \mathbb{R}$. Similar reasoning proves that $\check{H}^k_{\infty,\nabla}(T_\alpha,\mathbb{R}) \cong 0$ for $k > 2$. Thus we have proven the following.

Theorem 7.3. Let T_α denote the irrational torus, then

$$\check{H}^k_{\infty,\nabla}(T_\alpha, R) \cong \begin{cases} \mathbb{R}^2, & k = 1, \\ \mathbb{R}, & k = 2, \\ 0, & k > 2. \end{cases} \tag{78}$$

[18] Near the completion of this paper, we learned that an analogous exact sequence was also obtained in [Jaz21, Page 27] using completely different methods in the framework of homotopy type theory.

The reader should note that the above computations only work because the irrational torus has the property that its deRham cohomology is equal to its closed forms. This is not the case for general diffeological spaces, and therefore Corollary 7.2 is not generally helpful for computations. Therefore we desire an exact sequence which uses the deRham cohomology of a diffeological space rather than its closed forms.

Definition 7.4. Given a diffeological space X and $k \geq 1$, let $\check{H}^k_{\mathrm{conn}}(X,\mathbb{R})$ denote the subspace of $H^k_{\mathrm{dR}}(X) \oplus \check{H}^k_\infty(X,\mathbb{R})$ generated by the subset of pairs $([F],[g])$ where F is the curvature form $F = d\omega^k$ of a diffeological $\mathbb{R}$-bundle $(k-1)$-gerbe with connection $(\omega^k, \omega^{k-1}, \dots, \omega^1, g)$.

The vector space $\check{H}^k_{\mathrm{conn}}(X,\mathbb{R})$ sits in an exact sequence

$$0 \to \check{H}^k_{\infty,\mathrm{triv}}(X,\mathbb{R}^\delta) \to \check{H}^k_{\infty,\nabla}(X,\mathbb{R}) \to \check{H}^k_{\mathrm{conn}}(X,\mathbb{R}) \to 0, \tag{79}$$

where $\check{H}^k_{\infty,\mathrm{triv}}(X,\mathbb{R}^\delta) \subset \check{H}^k_\infty(X,\mathbb{R}^\delta)$ is the subspace of isomorphism classes of trivial $\mathbb{R}$-bundle $(k-1)$-gerbes with flat connection.

Let us now define a new sequence of vector spaces

$$\check{H}^k_\infty(X,\mathbb{R}^\delta) \xrightarrow{\alpha} \check{H}^k_{\mathrm{conn}}(X,\mathbb{R}) \xrightarrow{\beta} H^{k+1}_{\mathrm{dR}}(X) \xrightarrow{\gamma} \check{H}^{k+1}_\infty(X,\mathbb{R}^\delta).$$

The map α takes an isomorphism class of an $\mathbb{R}$-bundle k-gerbe with flat connection $[\omega^k,\dots,\omega^1,g]$ and gives $(0,[g])$. The map β sends $([F],[g]) \mapsto [F]$. Finally, γ sends $[\omega]$ to the isomorphism class of the $\mathbb{R}$-bundle k-gerbe with connection $(\omega, 0, \dots, 0)$. This map is well defined, because if $\omega' - \omega = d\tau$ for some $\tau \in \Omega^k(X)$, then $(\omega' - \omega, 0, \dots, 0) = D(\tau, 0, \dots, 0)$.

Theorem 7.5. Given a diffeological space X and $k \geq 1$, the sequence of vector spaces

$$\check{H}^k_\infty(X,\mathbb{R}^\delta) \xrightarrow{\alpha} \check{H}^k_{\mathrm{conn}}(X,\mathbb{R}) \xrightarrow{\beta} H^{k+1}_{\mathrm{dR}}(X) \xrightarrow{\gamma} \check{H}^{k+1}_\infty(X,\mathbb{R}^\delta) \tag{80}$$

is exact.

Proof. Note that $\beta\alpha = 0$. Suppose $([F],[g]) \in \check{H}^k_{\mathrm{conn}}(X,\mathbb{R})$ and consider $\gamma\beta([F],[g]) = [F,0,\dots,0]$. Since $([F],[g]) \in \check{H}^k_{\mathrm{conn}}(X,\mathbb{R})$, there exists a $\mathbb{R}$-bundle $(k-1)$-gerbe with connection $(\omega^k,\dots,\omega^0,g)$ such that

$$D(\omega^k,\dots,\omega^0,g) = (F,0,\dots,0).$$

Therefore $[F,0,\dots,0] = 0$. In fact, $\gamma\beta([F],[g]) = 0$ if and only if there exists an $\mathbb{R}$-bundle $(k-1)$-gerbe with connection such that the above equation holds. This implies that $\ker\gamma = \operatorname{im}\beta$.

Now suppose that $\beta([F],[g]) = [F] = 0$. Then there exists a global k-form τ such that $d\tau = F$. Since $([F],[g]) \in \check{H}^k_{\mathrm{conn}}(X,\mathbb{R})$, there exists an $\mathbb{R}$-bundle $(k-1)$-gerbe $(\omega^k,\dots,g)$ such that $d\omega^k = F$. Then $\omega^k - \tau \in \Omega^k(QX_0)$, $d(\omega^k - \tau) = 0$ and $\delta(\omega^k - \tau) = \delta\omega^k - \delta\tau = \delta\omega^k$. Therefore $(\omega^k - \tau, \omega^{k-1},\dots,\omega^0,g)$ defines an $\mathbb{R}$-bundle k-gerbe with flat connection. Thus $([F],[g]) = (0,[g]) \in \operatorname{im}\alpha$. □

We will refer to the exact sequence (80) as the **degree k PIZ exact sequence**. When $k = 1$, there is an interesting additional phenomenon.

Lemma 7.6. Let (A,g) and (A',g) be $\mathbb{R}$-bundle 0-gerbes/diffeological principal $\mathbb{R}$-bundles with connection on a diffeological space X with the same underlying $\mathbb{R}$-cocycle g, then dA and dA' are global closed 2-forms on X and their de Rham cohomology classes agree $[dA] = [dA']$.

Proof. Since (A,g) and (A',g) are both $\mathbb{R}$-bundle 0-gerbes with connection on X, this implies that for every map of plots f_0, we have

$$-(\delta A)_{f_0} = dg_{f_0} = -(\delta A')_{f_0}.$$

Now consider the form $A' - A \in \Omega^1(QX_0)$, defined plotwise by $(A' - A)_{p_0} = A'_{p_0} - A_{p_0}$. This is a global 1-form, because for every plot map f_0 we have

$$(\delta(A' - A))_{f_0} = (\delta A')_{f_0} - (\delta A)_{f_0} = 0.$$

Thus $A' - A \in \Omega^1(X)$. Similarly dA' and dA are also global 2-forms on X. Now $d(A'-A) \in \Omega^2(X)$ is an exact form, and $dA' - dA = d(A' - A)$. Thus dA' and dA represent the same de Rham cohomology class. □

Lemma 7.6 implies that $\check{H}^1_{\mathrm{conn}}(X, \mathbb{R})$ is isomorphic to the subspace of $\check{H}^1_\infty(X, \mathbb{R})$ generated by the subset of those diffeological principal $\mathbb{R}$-bundles that admit a connection, as every connection produces a unique cohomology class. Thus we see that $\check{H}^1_{\mathrm{conn}}(X, \mathbb{R})$ is exactly analogous to the term ${}^dE_2^{1,0}(X)$ in (71).

Now the degree k PIZ exact sequence (80) seems to be missing two terms compared to (71). These two terms reappear when $k = 1$, as we shall now prove.

Theorem 7.7. Given a diffeological space X, there exists a map $\theta : H^1_{\mathrm{dR}}(X) \to \check{H}^1_\infty(X, \mathbb{R}^\delta)$ such that the sequence of vector spaces

$$0 \to H^1_{\mathrm{dR}}(X) \xrightarrow{\theta} \check{H}^1_\infty(X, \mathbb{R}^\delta) \xrightarrow{\alpha} \check{H}^1_{\mathrm{conn}}(X, \mathbb{R}) \xrightarrow{\beta} H^2_{\mathrm{dR}}(X) \xrightarrow{\gamma} \check{H}^2_\infty(X, \mathbb{R}^\delta) \tag{81}$$

is exact.

Proof. The sequence is exact everywhere except for $H^1_{\mathrm{dR}}(X)$ and $\check{H}^1_\infty(X, \mathbb{R}^\delta)$ by Theorem 7.5. Now recall the isomorphism $\varphi : \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0]) \to \check{H}^0_\infty(X, [\mathbb{R} \to \Omega^1_{\mathrm{cl}}])$ induced by the map of presheaves of chain complexes described in Example 6.8 for $k = 1$. The map φ takes an isomorphism class of a diffeological principal $\mathbb{R}^\delta$-bundle cocycle $[g]$ and gives the isomorphism class of the $\mathbb{R}$-bundle 0-gerbe with connection $[0, g]$. Let $\theta : H^1_{\mathrm{dR}}(X) \to \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0])$ denote the map defined as follows. Let $[A] \in H^1_{\mathrm{dR}}(X)$ denote a cohomology class, and suppose that A is a global closed 1-form representing this class. Since it is closed, there exists an $a \in \mathbb{R}(QX_0)$ such that $da = A$. Then δa defines an $\mathbb{R}^\delta$ cocycle, as $d\delta a = \delta A = 0$. Let $\theta([A]) = [\delta a]$. This map is well defined, as suppose that $a, a' \in \mathbb{R}(QX_0)$ such that $da = da' = A$. Then $a - a'$ is a $\mathbb{R}^\delta$-coboundary between δa and $\delta a'$ as $d(a - a') = A - A = 0$ and $\delta a - \delta a' = \delta(a - a')$, so $[\delta a] = [\delta a']$.

We have a commutative diagram

$$\begin{tikzcd} H^1_{\mathrm{dR}}(X) \arrow[r, "\theta"] \arrow[dr, "\theta'"'] & \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0]) \arrow[r, "\alpha"] \arrow[d, "\varphi"] & \check{H}^1_{\mathrm{conn}}(X, \mathbb{R}) \\ & \check{H}^0_\infty(X, [\mathbb{R} \to \Omega^1_{\mathrm{cl}}]) & \end{tikzcd} \tag{82}$$

where α takes a $\mathbb{R}^\delta$-cocycle and considers it as a $\mathbb{R}$-cocycle, and $\theta'([A]) = [-A, 0]$. Now $\varphi\theta = \theta'$ because $\varphi\theta([A]) = [0, \delta a]$, $\theta'([A]) = [-A, 0]$, and $(-A, 0) - (0, \delta a) = (-A, -\delta a) = (-da, -\delta a) = D(-a)$.

Clearly $\alpha\theta = 0$. Let us show that $\operatorname{im}\theta = \ker\alpha$. Suppose that $[g]$ is the isomorphism class of a diffeological principal $\mathbb{R}^\delta$-bundle such that it is trivial as a diffeological principal $\mathbb{R}$-bundle. Then there exists a $\lambda \in \mathbb{R}(QX_0)$ such that $g = \delta\lambda$. Then $\theta([d\lambda]) = [g]$. Now let us show that θ is injective. It is enough to show that θ' is injective, as φ is an isomorphism. Suppose that $[A]$ and $[B]$ are cohomology classes such that $\theta'([A]) = [-A, 0] = [-B, 0] = \theta'([B])$. Then there exists a $\tau \in \mathbb{R}(QX_0)$ such that $(-A - (-B), 0) = (B - A, 0) = D\tau$, which implies that $\delta\tau = 0$, so that τ is a global 0-form and $d\tau = B - A$. Thus $[A] = [B]$. Thus we have proven that θ is injective. Now abuse notation and let $\check{H}^1_\infty(X, \mathbb{R}^\delta) = \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0])$. This proves that the above sequence is exact everywhere. □

Considering again the case where $X = T_\alpha$ is the irrational torus, from (74), (75) and (81), we obtain that

$$\check{H}^1_{\mathrm{conn}}(T_\alpha, \mathbb{R}) \cong \mathbb{R}, \tag{83}$$

which agrees with [Igl23]. From (79) and Theorem 7.3 we then obtain an isomorphism

$$\check{H}^1_{\infty,\mathrm{triv}}(T_\alpha, \mathbb{R}^\delta) \cong \mathbb{R}. \tag{84}$$

Similarly, from (80) we obtain an exact sequence

$$\mathbb{R} \to \check{H}^2_{\mathrm{conn}}(T_\alpha, \mathbb{R}) \to 0$$

so that $\check{H}^2_{\mathrm{conn}}(T_\alpha, \mathbb{R})$ is either 0 or $\mathbb{R}$.

Appendix A. Diffeological Principal Bundles with Connection

In this section we show that the notion of diffeological principal G-bundle with connection introduced in Example 6.4 is equivalent to Waldorf's, given in [Wal12, Definition 3.2.1].

Given a diffeological space X, and a Lie group G, recall the definition of the ∞-stack $\Omega^1(-,\mathfrak{g})//G$ from Example 6.4. The data of a map $QX \to \Omega^1(-,\mathfrak{g})//G$ is equivalent to a G-cocycle g and a collection $A = \{A_{p_0}\}$ of 1-forms $A_{p_0} \in \Omega^1(U_{p_0}, \mathfrak{g})$ satisfying

$$A_{p_1} = \mathrm{Ad}^{-1}_{g_{f_0}}(f_0^* A_{p_0}) + g^*_{f_0}\mathrm{mc}(G).$$

for every plot $p_0 : U_{p_0} \to X$. We refer to such a map $QX \to \Omega^1(-,\mathfrak{g})//G$ as a **G-cocycle with connection**.

Definition A.1. Let $\mathsf{Coc}_\nabla(X,G)$ denote the category whose objects are G-cocycles with connection on X, and whose morphisms $h : (A,g) \to (A',g')$ are collections $h = \{h_{p_0}\}$ of maps $h_{p_0} : U_{p_0} \to G$ such that h is a morphism of G-cocycles in the sense of Definition 2.6 and $A_{p_0} = \mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + h^*_{p_0}\mathrm{mc}(G)$ for every plot p_0 of X. It is easy to see that this category is a groupoid.

Definition A.2. Let $\pi : P \to X$ be a diffeological principal G-bundle where G is a Lie group. A **Waldorf connection** on P is a 1-form $\omega \in \Omega^1(P,\mathfrak{g})$ such that

$$\rho^*\omega = \mathrm{Ad}^{-1}_g(\mathrm{pr}^*\omega) + g^*\mathrm{mc}(G) \tag{85}$$

where $\rho : P\times G \to P$ is the action map, and $g : P\times G \to G$ and $\mathrm{pr} : P\times G \to P$ are the corresponding projection maps.

A morphism $f : (\omega, P) \to (\omega', P')$ of diffeological principal G-bundles on X with Waldorf connections is a morphism of diffeological principal G-bundles $f : P \to P'$ such that $f^*\omega' = \omega$. Such morphisms are isomorphisms. Given a diffeological space X, let $\mathsf{Wal}_G(X)$ denote the groupoid of diffeological principal G-bundles on X equipped with a Waldorf connection.

In [Min22, Section 3], we showed there is a functor $\mathsf{Cons} : \mathsf{Coc}(X,G) \to \mathsf{DiffPrin}_G(X)$ that takes a G-cocycle g and constructs a diffeological principal G-bundle $\mathsf{Cons}(g) = \pi : P \to X$ on X. Furthermore, by Theorem 2.7, this functor is an equivalence. Thus we need only understand how to construct a Waldorf connection from the collection $A = \{A_{p_0}\}$ of 1-forms and vice versa.

So let $g = \{g_{f_0}\}$ be a fixed G-cocycle representing a diffeological principal G-bundle $\mathsf{Cons}(g) = \pi : P \to X$. We wish to construct a 1-form ω on P from a G-cocycle with connection A on X. The diffeological principal G-bundle $\mathsf{Cons}(g)$ has a canonical plotwise trivialization $\varphi_{p_0} : U_{p_0} \times G \to p_0^*P$ such that if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots, then the induced map $\widetilde{f_0} : U_{p_1} \times G \to U_{p_0} \times G$ is given by $\widetilde{f_0}(x_{p_1}, h) = (f_0(x_{p_1}), g_{f_0}(x_{p_1}) \cdot h)$, where $g_{f_0} : U_{p_1} \to G$ is the component of the G-cocycle on f_0. See [Min22, Section 3] for more details.

Now let $q_0 : U_{q_0} \to P$ be a plot. We obtain a commutative diagram

$$\begin{tikzcd}
U_{q_0} \arrow[drrr, bend left, "q_0"] \arrow[dr, "k_{q_0}"] \arrow[ddrr, bend right, "1_{U_{q_0}}"'] & & & \\
& U_{q_0}\times G \arrow[r, "\varphi_{p_0}"] \arrow[dr] & p_0^*P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\pi"] \\
& & U_{q_0} \arrow[r, "p_0"'] & X
\end{tikzcd} \tag{86}$$

where $p_0 = \pi q_0$ and $k_{q_0} : U_{q_0} \to U_{q_0} \times G$ is the unique map given by the universal property of the pullback $U_{q_0} \times G \cong p_0^*P$. Since this map is over U_{q_0}, we have $k_{q_0}(x_{q_0}) = (x_{q_0}, g_{q_0}(x_{q_0}))$ for a unique map $g_{q_0} : U_{q_0} \to G$.

Now if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots, we obtain a commutative diagram

$$\begin{tikzcd} U_{q_1} \arrow[rr, "f_0"] \arrow[d, "k_{q_1}"'] & & U_{q_0} \arrow[d, "k_{q_0}"'] \\ U_{q_1} \times G \arrow[rr, "\widetilde{f_0}"] \arrow[dr] & & U_{q_0} \times G \arrow[dl] \\ & P & \end{tikzcd} \tag{87}$$

which implies that if $x_{q_1} \in U_{q_1}$, then

$$\widetilde{f_0} k_{q_1}(x_{q_1}) = \widetilde{f_0}(x_{q_1}, g_{q_1}(x_{q_1})) = (f_0(x_{q_1}), g_{f_0}(x_{q_1}) \cdot g_{q_1}(x_{q_1})) = (f_0(x_{q_1}), g_{q_0}(f_0(x_{q_1})) = k_{q_0} f_0(x_{q_1}).$$

From this we obtain the equation

$$g_{f_0} \cdot g_{q_1} = (g_{q_0} \circ f_0). \tag{88}$$

Now suppose that $A = \{A_{p_0}\}$ is a G-cocycle with connection for the fixed cocycle g. We wish to obtain a 1-form on P. Since P is a diffeological space, we can define it plotwise. Given a plot $q_0 : U_{q_0} \to P$, we obtain a plot $p_0 : U_{q_0} \to X$ of the base X by setting $p_0 = \pi q_0$. Thus there is a 1-form $A_{p_0} \in \Omega^1(U_{q_0}, \mathfrak{g})$ from the G-cocycle with connection. Now let

$$B_{q_0} = \mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + g^*_{q_0}\mathrm{mc}(G). \tag{89}$$

Thus $B_{q_0} \in \Omega^1(U_{q_0}, \mathfrak{g})$. We wish to show that this defines a 1-form on P, namely we need to check that if $f_0 : U_{q_1} \to U_{q_0}$ is a map of plots of P, then

$$f_0^* B_{q_0} = B_{q_1}. \tag{90}$$

So let $f_0 : U_{q_1} \to U_{q_0}$ be such a map of plots. Then we have

$$\begin{aligned} f_0^* B_{q_0} &= f_0^*\left(\mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + g^*_{q_0}\mathrm{mc}(G)\right) \\ &= \mathrm{Ad}^{-1}_{(g_{q_0}\circ f_0)}(f_0^* A_{p_0}) + (g_{q_0} \circ f_0)^*\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_1}}\mathrm{Ad}^{-1}_{g_{f_0}}(f_0^* A_{p_0}) + (g_{f_0} \cdot g_{q_1})^*\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_1}}\mathrm{Ad}^{-1}_{g_{f_0}}(f_0^* A_{p_0}) + \mathrm{Ad}^{-1}_{g_{q_1}}(g^*_{f_0}\mathrm{mc}(G)) + g^*_{q_1}\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_1}}\left(\mathrm{Ad}^{-1}_{g_{f_0}}(f_0^* A_{p_0}) + g^*_{f_0}\mathrm{mc}(G)\right) + g^*_{q_1}\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_1}}(A_{p_1}) + g^*_{q_1}\mathrm{mc}(G) \\ &= B_{q_1}. \end{aligned} \tag{91}$$

We have used the product rule for the Maurer-Cartan form

$$(g \cdot h)^*\mathrm{mc}(G) = \mathrm{Ad}^{-1}_h(g^*\mathrm{mc}(G)) + h^*\mathrm{mc}(G), \tag{92}$$

on the fourth line above, which can easily be verified using the description of $\mathrm{mc}(G)$ as $g^{-1}dg$.

Thus the collection $\{B_{q_0}\}$ defines a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with $\omega_{q_0} = B_{q_0}$. We must still show that ω is a Waldorf connection.

We will check the equation (85) plotwise on $P \times G$. A plot of $P \times G$ is a pair of plots $q_0 : U_{q_0} \to P$ and $h_0 : U_{q_0} \to G$, which we shall pair to form the plot $\langle q_0, h_0\rangle : U_{q_0} \to P \times G$. Let us examine $(\rho^*\omega)_{\langle q_0, h_0\rangle}$. This is the 1-form $\omega_{\rho\langle q_0, h_0\rangle}$, where $\rho : P \times G \to P$ is the action map. We can thus write $\rho\langle q_0, h_0\rangle = q_0 \cdot h_0$, where $\cdot$ is the action of G on P. Thus we wish to compute $\omega_{q_0 \cdot h_0}$. Looking plotwise, it is easy to see that

$$g_{q_0 \cdot h_0} = g_{q_0} \cdot h_0. \tag{93}$$

Thus we have

$$
\begin{aligned}
\omega_{q_0\cdot h_0} &= B_{q_0\cdot h_0} \\
&= \mathrm{Ad}^{-1}_{g_{q_0\cdot h_0}}(A_{p_0}) + g^*_{q_0\cdot h_0}\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{h_0}\mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + (g_{q_0}\cdot h_0)^*\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{h_0}\mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + \mathrm{Ad}^{-1}_{h_0} g^*_{q_0}\mathrm{mc}(G) + h_0^*\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{h_0}(B_{q_0}) + h_0^*\mathrm{mc}(G).
\end{aligned} \tag{94}
$$

Pulling back to $P \times G$ gives precisely the equation (85). So given a G-cocycle with connection (A,g), let $\mathsf{Cons}_\nabla(A,g) = (\omega,P)$ denote the diffeological principal G-bundle $P = \mathsf{Cons}(g)$ equipped with Waldorf connection ω.

Now suppose that $h : (A,g) \to (A',g')$ is a morphism of G-cocycles with connection on X. We wish to obtain a morphism of diffeological principal G-bundles that preserve the Waldorf connection. By [Min22, Section 3], we know that $\mathsf{Cons}(h) : \mathsf{Cons}(g) \to \mathsf{Cons}(g')$ is a map of the respective diffeological principal G-bundles. We need only show that $\mathsf{Cons}(h)$ preserves the Waldorf connection. Let $(\omega,P) = \mathsf{Cons}_\nabla(A,g)$ and $(\omega',P') = \mathsf{Cons}_\nabla(A',g')$, and let $\widetilde{h} = \mathsf{Cons}(h)$ denote the corresponding morphism given by the morphism h of G-cocycles. For every plot $q_0 : U_{q_0} \to P$ we obtain the following commutative diagram

$$
\begin{tikzcd}
& U_{q_0} \arrow[dl, "k_{q_0}"'] \arrow[dr, "k'_{\widetilde{h}q_0}"] & \\
U_{q_0}\times G \arrow[rr, "\widetilde{h_{q_0}}"] \arrow[d, "l_{q_0}"'] & & U_{q_0}\times G \arrow[d, "l'_{\widetilde{h}q_0}"] \\
P \arrow[rr, "\widetilde{h}"'] & & P'
\end{tikzcd} \tag{95}
$$

where $\widetilde{h_{q_0}}(x_{q_0},g) = (x_{q_0}, h_{q_0}(x_{q_0})\cdot g)$ for $h_{q_0} : U_{q_0} \to G$ the component of the morphism h of cocycles. The above diagram also implies that

$$
k'_{\widetilde{h}q_0}(x_{q_0}) = (x_{q_0}, g'_{\widetilde{h}q_0}(x_{q_0})) = (x_{q_0}, h_{p_0}(x_{q_0})\cdot g_{q_0}(x_{q_0})) = \widetilde{h_{q_0}}k_{q_0}(x_{q_0}), \tag{96}
$$

and thus we have

$$
g'_{\widetilde{h}q_0} = h_{q_0}\cdot g_{q_0}. \tag{97}
$$

We wish to show that $\widetilde{h}^*\omega' = \omega$. It is therefore equivalent to show that

$$
(\widetilde{h}^*\omega')_{q_0} = \omega'_{\widetilde{h}q_0} = B'_{\widetilde{h}q_0} = B_{q_0} = \omega_{q_0}. \tag{98}
$$

Now we have

$$
\begin{aligned}
B'_{\widetilde{h}q_0} &= \mathrm{Ad}^{-1}_{g_{\widetilde{h}q_0}}(A'_{p_0}) + g'^*_{\widetilde{h}q_0}\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{g_{q_0}}\mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + (h_{p_0}\cdot g_{q_0})^*\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{g_{q_0}}\mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + \mathrm{Ad}^{-1}_{g_{q_0}}(h^*_{p_0}\mathrm{mc}(G)) + g^*_{q_0}\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{g_{q_0}}(\mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + h^*_{p_0}\mathrm{mc}(G)) + g^*_{q_0}\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + g^*_{q_0}\mathrm{mc}(G) \\
&= B_{q_0}.
\end{aligned} \tag{99}
$$

Thus $\widetilde{h} : P \to P'$ preserves the Waldorf connections. In summary, we have constructed a functor $\mathsf{Cons}_\nabla : \mathsf{Coc}(X,G) \to \mathsf{Wal}_G(X)$. Now we wish to show that this functor is an equivalence of groupoids.

Now let us show that if we have a Waldorf connection ω on P, we can obtain an G-cocycle with connection. Suppose that $\pi : P \to X$ is a diffeological principal G-bundle, and choose a fixed plotwise trivialization φ. From this we obtain a G-cocycle g. Suppose that $\omega \in \Omega^1(P,\mathfrak{g})$ is a Waldorf connection, and let $p_0 : U_{p_0} \to X$ be a plot. We obtain the commutative diagram

$$\begin{tikzcd}
U_{p_0} \times G \arrow[r, "\varphi_{p_0}"] \arrow[dr] & p_0^*P \arrow[r, "\psi_{p_0}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P \arrow[d, "\pi"] \\
& U_{p_0} \arrow[r, "p_0"'] \arrow[ul, bend left, "\sigma_{p_0}"] & X
\end{tikzcd}$$

where φ_{p_0} is the fixed trivialization, which is a G-equivariant diffeomorphism over U_{p_0}, and $\sigma_{p_0} : U_{p_0} \to U_{p_0} \times G$ is the canonical section $\sigma_{p_0}(x_{p_0}) = (x_{p_0}, e_G)$. Let $q_0 : U_{p_0} \to P$ be given by $q_0 = \psi_{p_0}\varphi_{p_0}\sigma_{p_0}$. Suppose that $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots of X. Then we obtain a diagram

$$\begin{tikzcd}
U_{p_1} \arrow[rr, "f_0"] \arrow[d, "\sigma_{p_1}"'] & & U_{p_0} \arrow[d, "\sigma_{p_0}"] \\
U_{p_1} \times G \arrow[rr, "\widetilde{f_0}"] \arrow[d, "\varphi_{p_1}"'] & & U_{p_0} \times G \arrow[d, "\varphi_{p_0}"] \\
p_1^*P \arrow[rr, "\widehat{f_0}"] \arrow[dr, "\psi_{p_1}"'] & & p_0^*P \arrow[dl, "\psi_{p_0}"] \\
& P &
\end{tikzcd}$$

Notice that the middle square and the bottom triangle commute, but the top square does not commute, as $\widetilde{f_0}\sigma_{p_1}(x_{p_1}) = (f_0(x_{p_1}), g_{f_0}(x_{p_1}))$ while $\sigma_{p_0}(f_0(x_{p_1})) = (f_0(x_{p_1}), e_G)$. Thus we have

$$\begin{aligned}
q_1(x_{p_1}) &= (\psi_{p_1}\varphi_{p_1}\sigma_{p_1})(x_{p_1}) \\
&= \psi_{p_0}\varphi_{p_0}\widetilde{f_0}\sigma_{p_1}(x_{p_1}) \\
&= \psi_{p_0}\varphi_{p_0}(f_0(x_{p_1}), g_{f_0}(x_{p_1})) \\
&= (\psi_{p_0}\varphi_{p_0})\big[(f_0(x_{p_1}), e_G)\cdot g_{f_0}(x_{p_1})\big] \\
&= (\psi_{p_0}\varphi_{p_0}\sigma_{p_0}f_0)(x_{p_1})\cdot g_{f_0}(x_{p_1}) \\
&= (q_0 \circ f_0)(x_{p_1})\cdot g_{f_0}(x_{p_1}).
\end{aligned} \tag{100}$$

where on the fourth line we used the fact that φ_{p_0} and ψ_{p_0} are G-equivariant.

Thus we have obtained the equation

$$q_1 = (q_0 \circ f_0)\cdot g_{f_0}. \tag{101}$$

Now let $A_{p_0} = \omega_{q_0}$. Note that $f_0^*A_{p_0} \neq A_{p_1}$ since f_0 is not a map of plots from q_1 and q_0, i.e. $(q_0 \circ f_0) \neq q_1$. Consider the equation (85) at the plot $\langle (q_0 \circ f_0), g_{f_0}\rangle : U_{q_1} \to P \times G$. Note that

$$(\rho^*\omega)_{\langle (q_0\circ f_0), g_{f_0}\rangle} = \omega_{\rho\langle (q_0\circ f_0), g_{f_0}\rangle} = \omega_{(q_0\circ f_0)\cdot g_{f_0}} = \omega_{q_1} = A_{p_1}, \tag{102}$$

and

$$\mathrm{Ad}^{-1}_{g_{f_0}}((\mathrm{pr}^*\omega)_{\langle (q_0\circ f_0), g_{f_0}\rangle}) + g_{f_0}^*\mathrm{mc}(G) = \mathrm{Ad}^{-1}_{g_{f_0}}(\omega_{(q_0\circ f_0)}) + g_{f_0}^*\mathrm{mc}(G). \tag{103}$$

Now $f_0^*A_{p_0} = f_0^*\omega_{q_0} = \omega_{(q_0\circ f_0)}$ because f_0 is a plot map from q_0 to $(q_0 \circ f_0)$ trivially, and ω is a 1-form on P. Thus we obtain equation (60), so the collection $A = \{A_{p_0}\}$ defines a G-cocycle with connection on X.

Now if $\widetilde{h} : (\omega, P) \to (\omega', P')$ is a map of diffeological principal G-bundles with Waldorf connection, we want to show that it induces a map of G-cocycle with connection. We know that $\widetilde{h}$ induces a map h of the G-cocycles g and g' representing P and P' respectively, and we wish to show that $A_{p_0} = \mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + h_{p_0}^*\mathrm{mc}(G)$ for every plot $p_0 : U_{p_0} \to X$. We know that $\widetilde{h}^*\omega' = \omega$, which is equivalent to asking that $B'_{\widetilde{hq_0}} = B_{q_0}$. So if p_0 is a plot of X, then we obtain a plot $q_0 : U_{p_0} \to P$ in the same way as above. We obtain

$$\begin{aligned} A_{p_0} &= \mathrm{Ad}_{g_{q_0}}(B_{q_0} - g_{q_0}^*\mathrm{mc}(G)) \\ &= \mathrm{Ad}_{g_{q_0}}(B'_{\widetilde{hq_0}} - g_{q_0}^*\mathrm{mc}(G)) \\ &= \mathrm{Ad}_{g_{q_0}}[\mathrm{Ad}_{g_{\widetilde{hq_0}}}(A'_{p_0}) + g_{\widetilde{hq_0}}^*\mathrm{mc}(G) - g_{q_0}^*\mathrm{mc}(G)] \\ &= \mathrm{Ad}_{g_{q_0}}\mathrm{Ad}_{g_{q_0}}^{-1}\mathrm{Ad}_{h_{p_0}}^{-1}(A'_{p_0}) + \mathrm{Ad}_{g_{q_0}}\mathrm{Ad}_{g_{q_0}}^{-1}(h_{p_0}^*\mathrm{mc}(G)) \\ &+ \mathrm{Ad}_{g_{q_0}}(g_{q_0}^*\mathrm{mc}(G)) - \mathrm{Ad}_{g_{q_0}}(g_{q_0}^*\mathrm{mc}(G)) \\ &= \mathrm{Ad}_{h_{p_0}}^{-1}(A'_{p_0}) + h_{p_0}^*\mathrm{mc}(G), \end{aligned} \tag{104}$$

where we have basically done the computation of (99) in reverse. Thus h is a morphism of G-cocycles with connection.

Theorem A.3. Given a diffeological space X and a Lie group G, the functor

$$\mathsf{Cons}_\nabla : \mathsf{Coc}_\nabla(X,G) \to \mathsf{Wal}_G(X), \tag{105}$$

is an equivalence of groupoids.

Proof. This follows from combining [Min22, Theorem 5.13] with the above constructions. □

Remark A.4. It should be said that when $G = \mathbb{R}$ or $G = S^1$, one can check that a Waldorf connection reduces to a connection 1-form in the sense of [Igl23, Section 5.3], thus we have an equivalence between all three definitions of diffeological principal G-bundle with connection in these cases.

Remark A.5. There is nothing stopping one from extending the above definition to the case when G is a diffeological group. In this case then $\mathfrak{g}$ should be the internal tangent space [CW15] to the diffeological group G at the identity. Nothing in this Appendix depended on G being a Lie group, so the whole previous discussion extends to this case. It is an interesting question to see how far one can go with this analogy. For instance, does this extended definition agree with that given in [Igl13, Article 8.32]? We leave this question for future work.

Appendix B. Totalization

Given a presheaf of chain complexes A and a diffeological space X, we wish to compute the ∞-stack cohomology of X with values in A. This is defined as the abelian group

$$\check{H}^0_\infty(X,A) := \pi_0 \mathbb{R}\check{\mathbb{H}}(X,A). \tag{106}$$

We will use the Dold-Kan correspondence to get an amenable model for the homotopy type of $\mathbb{R}\check{\mathbb{H}}(X,A)$.

Let C be a cosimplicial chain complex, whose cosimplicial degree is denoted by the chain complex C^p. The qth degree of the chain complex C^p is denoted $C^{p,q}$, with differential $d : C^{p,q} \to C^{p,q-1}$. From a cosimplicial chain complex we can obtain a (mixed) double complex by applying the dual of the Dold-Kan correspondence [Min22, Section 4.3] to $C^\bullet$ to obtain a $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-graded vector space with two operators $d : C^{p,q} \to C^{p,q-1}$ given by the differential of each C^p and $\delta : C^{p,q} \to C^{p+1,q}$ defined as the alternating sum $\sum_{i=0}^p (-1)^i d^i$ of the coface maps of C, with the property that $d\delta = \delta d$. From this we can obtain an unbounded ($\mathbb{Z}$-graded) chain complex $K = \mathrm{tot}^{\mathbb{Z}} C$, with

$$(\mathrm{tot}^{\mathbb{Z}} C)_k = \prod_{q-p=k} C^{p,q} \tag{107}$$

and differential $D = (d - (-1)^{q-p}\delta)$. In order to obtain a non-negatively graded chain complex, we apply smart truncation to obtain $\mathrm{tot}\, C = \tau_{\geq 0} \mathrm{tot}^{\mathbb{Z}} C$.

Proposition B.1. Given a cosimplicial chain complex C, we have the following isomorphism of chain complexes,

$$\int_{n\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}[\Delta^n], C^n) \cong \operatorname{tot} C, \tag{108}$$

where $\underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}[\Delta^n], C^n)$ is the mapping chain complex defined by (55).

Proof. For the rest of this proof only, let $\Delta^n = N\mathbb{R}\Delta^n$. The chain complex $E := \int_{n\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(\Delta^n, C^n)$ is isomorphic to the equalizer

$$\operatorname{eq}\left(\prod_{[n]\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(\Delta^n, C^n) \rightrightarrows \prod_{f:[m]\to[n]} \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(\Delta^m, C^n) \right). \tag{109}$$

It is equipped with the usual differential of mapping chain complexes, namely $d_E : E_k \to E_{k-1}$ is the map

$$d_E(\varphi) = d_{C^\bullet} \circ \varphi - (-1)^k \varphi \circ d_{\Delta^\bullet}. \tag{110}$$

Thus for $k \in \mathbb{Z}$, an element $\varphi \in E_k$ consists of a family of degree k maps $\varphi_n : \Delta^n \to C^n$, such that for every map $f : [m] \to [n]$ the following diagram commutes

$$\begin{tikzcd} \Delta^m \arrow[r, "\varphi_m"] \arrow[d, "f"'] & C^m \arrow[d, "C^f"] \\ \Delta^n \arrow[r, "\varphi_n"'] & C^n \end{tikzcd} \tag{111}$$

and this makes sense, as pre or post-composing a degree k map of chain complexes with a chain map is a degree k map. This is equivalent to having a commutative diagram of the form

$$\begin{tikzcd} \Delta^0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\varphi_0"'] & \Delta^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\varphi_1"'] & \Delta^2 \arrow[r, shift left=3] \arrow[r, shift left] \arrow[r, shift right] \arrow[r, shift right=3] \arrow[d, "\varphi_2"'] & \cdots \\ C^0 \arrow[r, shift left] \arrow[r, shift right] & C^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & C^2 \arrow[r, shift left=3] \arrow[r, shift left] \arrow[r, shift right] \arrow[r, shift right=3] & \cdots \end{tikzcd} \tag{112}$$

where we have hidden the codegeneracy maps for clarity. For each $n \geq 0$, a degree k map $\varphi_n : \Delta^n \to C^n$ is equivalently the data of an element x_n in degree $k+n$ in C^n, corresponding to the top non-degenerate n-simplex $\iota_n \in (\Delta^n)_n$, along with an element $x_n \circ f$ in degree $k+m$ for every map $f : [m] \to [n]$. However, the diagram commuting implies that $x_n \circ f = C^f x_m$. In other words, the data of the $\{x_n\}_{n\geq 0}$ completely determine the whole diagram. Thus for $k \in \mathbb{Z}$, there is a bijection $E_k \cong \left(\operatorname{tot}^{\mathbb{Z}} C\right)_k \cong \prod_{q-p=k} C^{p,q}$. Furthermore their differentials agree, thus defining an isomorphism $E \cong \operatorname{tot}^{\mathbb{Z}} C$. Since $\int_{n\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}[\Delta^n], C^n) = \tau_{\geq 0} E$ and $\operatorname{tot} C = \tau_{\geq 0} \operatorname{tot}^{\mathbb{Z}} C$, they are isomorphic. □

Remark B.2. Let d_{Map} and d_v denote the differentials $\prod_{q-p=k} C^{p,q} \to \prod_{q-p=k-1} C^{p,q}$ defined componentwise by

$$d_{\mathrm{Map}} = (d - (-1)^{q-p}\delta), \qquad d_v = (d + (-1)^q \delta).$$

The differential d_v is more commonly seen for total complexes in the literature. There is an isomorphism $(\operatorname{tot} C, d_{\mathrm{Map}}) \cong (\operatorname{tot} C, d_v)$ given as follows. We wish to find isomorphisms $\psi_k : (\operatorname{tot} C)_k \to (\operatorname{tot} C)_k$ making the following diagrams commute for all $k \geq 0$

$$\begin{tikzcd} \prod_{q-p=k} C^{p,q} \arrow[r, "\psi_k"] \arrow[d, "d_{\mathrm{Map}}"'] & \prod_{q-p=k} C^{p,q} \arrow[d, "d_v"] \\ \prod_{q-p=k-1} C^{p,q} \arrow[r, "\psi_{k-1}"'] & \prod_{q-p=k-1} C^{p,q} \end{tikzcd}$$

namely we want an isomorphism of chain complexes. Let us define maps $\sigma_{p,q} : C^{p,q} \to C^{p,q}$ by

$$\sigma_{p,q} = \begin{cases} \mathrm{id} & \text{if } p \equiv 0,3 \pmod 4 \\ -\mathrm{id} & \text{if } p \equiv 1,2 \pmod 4). \end{cases}$$

Then set $\psi_k = \prod_{q-p=k} \sigma_{p,q}$. This gives the desired isomorphism[19].

Let us now examine how Proposition B.1 helps us compute ∞-stack cohomology for diffeological spaces. Suppose that A' is a presheaf of chain complexes such that $A = \mathsf{DK}A'$ is an ∞-stack, and X is a diffeological space. Then the 0th ∞-stack cohomology of X with values in A is given by $\pi_0 \check{\mathbb{H}}(X,A)$. Let us compute $\check{\mathbb{H}}(X,A)$.

$$\begin{aligned} \check{\mathbb{H}}(X,A) &= \underline{\mathsf{sPre}(\mathsf{Cart})}(QX, \mathsf{DK}A') \\ &\cong \underline{\mathsf{sPre}(\mathsf{Cart})}\left(\int^n \coprod_{(f_{n-1},\dots,f_0)} yU_{p_n} \times \Delta^n, \mathsf{DK}A' \right) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \underline{\mathsf{sSet}}(\Delta^n, \underline{\mathsf{sPre}(\mathsf{Cart})}(yU_{p_n}, \mathsf{DK}A')) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \underline{\mathsf{sSet}}(\Delta^n, [\mathsf{DK}A'](U_{p_n})) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \underline{\mathsf{Ch}}(N\mathbb{R}\Delta^n, A'(U_{p_n})) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \mathsf{DK}\,\underline{\mathsf{Map}_{\mathsf{Ch}}}(N\mathbb{R}\Delta^n, A'(U_{p_n})) \\ &\cong \mathsf{DK} \int_n \underline{\mathsf{Map}_{\mathsf{Ch}}}(N\mathbb{R}\Delta^n, \prod_{(f_{n-1},\dots,f_0)} A'(U_{p_n})) \\ &\cong \mathsf{DK}\, \mathsf{tot}\, A'(QX), \end{aligned} \tag{113}$$

where the last isomorphism follows from Proposition B.1, and the third to last isomorphism follows from Lemma 5.7. Thus we have proven the following.

Proposition B.3. Given a presheaf of chain complexes A' such that $A = \mathsf{DK}A'$ is an ∞-stack, and X a diffeological space, the 0th ∞-stack cohomology of X with values in A is given by

$$\check{H}^0_\infty(X,A) \cong H_0(\mathsf{tot}\,A'(QX)).$$

Propositon B.3 allows us to get a component level description of ∞-stack cohomology of diffeological spaces with values in the ∞-stacks of interest, see Section 6. Let us now use Proposition B.1 to prove the following well-known folklore result.

Proposition B.4. Let C be a cosimplicial chain complex, then

$$\mathrm{holim}_{n\in\Delta} C^n \simeq \mathsf{tot}\, C, \tag{114}$$

where we are computing the homotopy limit in the category of chain complexes equipped with the projective model structure.

Proof. First let us show that every cosimplicial chain complex C is Reedy fibrant. For more information about Reedy categories, see [Rie14, Section 14]. We wish to show that the matching map $C^n \to M^n C$ is a projective fibration. To do so, it will be sufficient to show that if A is a cosimplicial vector space, then the canonical map $s : A^n \to M^n A$, defined by the same limit above, is surjective. This is sufficient because limits of chain complexes are computed degreewise, and a map is a projective fibration if and only if it is surjective in all positive degrees.

[19]We obtained the maps $\sigma_{p,q}$ by carefully following the procedure outlined in [Ric].

We follow the proof[20] given in [Jar, Lemma 21.1]. Let D_n denote the category whose objects are surjective maps $[n] \xrightarrow{\sigma} [k]$ where $k = n-1$ or $k = n-2$, and whose morphisms are either identities or coface maps $s^j : [n-1] \to [n-2]$

$$\begin{tikzcd} & {[n]} \arrow[dl, two heads, "s^i"'] \arrow[dr, two heads, "s^j s^i"] & \\ {[n-1]} \arrow[rr, two heads, "s^j"] & & {[n-2]} \end{tikzcd}$$

Then by [Hir09, Proposition 15.2.6],

$$M^n A \cong \lim_{\sigma : [n] \twoheadrightarrow [k]} A^k. \tag{115}$$

Now let us label every object of D_n by either $s^i : [n] \to [n-1]$ or $\sigma : [n] \to [n-2]$, and every non-identity morphism by a pair (s^i, s^j). Since $s^j s^i = s^i s^{j+1}$ for every $i \leq j$, the objects $s^j s^i : [n] \to [n-2]$ and $s^i s^{j+1} : [n] \to [n-2]$ are the same, but the morphisms (s^i, s^j) and (s^{j+1}, s^i) are not. We can write the above limit as the equalizer (where we are not denoting the identity maps)

$$M^n A \cong \mathrm{eq}\left(\prod_{s^i} A^{n-1} \times \prod_{\sigma} A^{n-2} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{(s^i, s^j)} A^{n-2} \right), \tag{116}$$

where α is defined in component (s^i, s^j) by $\alpha(a, a') = s^j a_i$, and β is defined in component (s^i, s^j) by $\beta(a, a') = a'_{s^j s^i}$. Since β in component (s^i, s^j) and in component (s^{j+1}, s^i) are equal $a'_{s^j s^i} = a'_{s^i s^{j+1}}$ for $i \leq j$, this equalizer will be isomorphic to the subspace of $(A^{n-1})^n$ of those tuples $a = (a_0, \dots, a_{n-1})$ where $s^j a_i = s^i a_{j+1}$ for $i \geq j$. The matching map $s : A^n \to M^n A$ is then given by $s(a_0, \dots, a_{n-1}) = (s^0 a_0, \dots, s^{n-1} a_{n-1})$.

Now let us prove by induction that s is surjective. In the base case, note that $(0, \dots, 0) = s(0, \dots, 0)$. Now suppose that every element $b \in M^n A$ of the form $b = (b_0, \dots, b_{j-1}, 0, \dots, 0)$ is in the image of s. We wish to show that every element of the form $a = (a_0, \dots, a_{j-1}, a_j, 0, \dots, 0)$ is in the image of s.

First note that for $i \leq j$, we have $s^j a_i = s^i a_{j+1} = s^i 0 = 0$. Thus $s^j d^i a_i = d^i s^{j-1} a_i = 0$ for $i < j$ and $s^j d^j a_j = a_j$. Thus

$$a - s(d^j a_j) = (a_0 - s^0 d^j a_j, \dots, a_{j-1} - s^{j-1} d^j a_j, 0, \dots, 0).$$

By the induction hypothesis, there exists a $c \in A^n$ such that $s(c) = a - s(d^j a_j)$. Therefore $a = s(c + d^j a_j)$.

So we have shown that $s : A^n \to M^n A$ is surjective. This implies that $s : C^n \to M^n C$ is a projective fibration for all cosimplicial chain complexes C. This implies that all cosimplicial chain complexes are Reedy fibrant.

Since C is Reedy fibrant, [Hir09, Theorem 19.8.7] implies that the totalization of C computes the homotopy limit, i.e. $\mathrm{holim}_\Delta C \simeq \int_{n \in \Delta} \underline{\mathrm{Map}_{\mathrm{Ch}}}(N\mathbb{R}[\Delta^n], C^n)$. Thus Proposition B.1 proves that $\mathrm{holim}_\Delta C \simeq \mathrm{tot}\, C$. □

Remark B.5. During the writing of this paper, the preprint [Ara23] came out, which also proves Proposition B.4 in greater generality. However since the scope of our argument is much smaller, we believe our proof of Proposition B.4 is simpler and more direct.

Appendix C. Proof of Theorem 7.1

In this section we prove Theorem 7.1. We will need several technical preliminary results.

[20] Note that the proof given in that note has several typographical errors, which is why we chose to reproduce a full proof here.

Given a chain complex C, consider the chain complex $C^{\Delta^1} \cong \underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}\Delta^1, C)$. This is the chain complex with $C_n^{\Delta^1} = C_n \oplus C_n \oplus C_{n+1}$ for $n > 0$, and with differential

$$d_n : C_n \oplus C_n \oplus C_{n+1} \to C_{n-1} \oplus C_{n-1} \oplus C_n$$

given by $d_n(x,y,z) = (dx, dy, dz - (-1)^n[-x+y])$.

This means that for $k = 0$, we have

$$C_0^{\Delta^1} \cong \ker\left(C_0 \oplus C_0 \oplus C_1 \xrightarrow{d_0} 0 \oplus 0 \oplus C_0\right)$$

where $d_0(x,y,z) = (0,0,dz+x-y)$. There is an isomorphism $C_0^{\Delta^1} \cong C_0 \oplus C_1$ given by the maps

$$\sigma : C_0^{\Delta^1} \to C_0 \oplus C_1, \qquad \sigma(x,y,z) = (x,z)$$

$$\tau : C_0 \oplus C_1 \to C_0^{\Delta^1}, \qquad \tau(x,z) = (x, x+dz, z).$$

Thus the differential $d : C_1^{\Delta^1} \to C_0^{\Delta^1}$ is isomorphic to the map $\alpha = \sigma \circ d_1$,

$$\alpha : C_1 \oplus C_1 \oplus C_2 \to C_0 \oplus C_1, \qquad \alpha(x,y,z) = \sigma d_1(x,y,z) = \sigma(dx, dy, dz - x + y) = (dx, dz - x + y).$$

The map $\pi : C^{\Delta^1} \to C \oplus C$ is given in degree $k > 0$ by

(117) $$\pi_k : C_k \oplus C_k \oplus C_{k+1} \to C_k \oplus C_k, \qquad \pi_k(x,y,z) = (x,y).$$

It is given in degree $k = 0$ by

(118) $$\pi_0 : C_0 \oplus C_1 \to C_0 \oplus C_0, \qquad \pi_0(x,z) = (x, x+dz).$$

Let us now state a few model categorical results that we will need for the proof of Theorem 7.1.

Lemma C.1 ([Hir09, Corollary 13.3.8])**.** Let $\mathcal{C}$ be a right proper model category and let

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

be a pullback square in $\mathcal{C}$ such that at least one of maps f or g is a fibration. Then the above square is a homotopy pullback square.

Lemma C.2 ([Hir09, Proposition 13.3.15])**.** Let $\mathcal{C}$ be a right proper model category, and suppose we have a commutative diagram of the form

(119) $$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

and suppose that the right hand square is a homotopy pullback square. Then the left hand square is a homotopy pullback square if and only if the outer rectangle is a homotopy pullback square.

Lemma C.3 ([Sch13, Corollary 2.3.10])**.** Let $\mathcal{C}$ be a model category, and suppose X, Y, Z are fibrant objects in $\mathcal{C}$ and $f : X \to Z$ and $g : Y \to Z$ are maps between them. Then the homotopy pullback of f and g is naturally weak equivalent to the actual pullback

(120) $$\begin{tikzcd} X \times_Z^h Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z^I \arrow[d] \\ X \times Y \arrow[r, "f \times g"'] & Z \times Z \end{tikzcd}$$

where $Z^I \to Z \times Z$ is a path object for Z.

Lemma C.4 ([Hir09, Proposition 3.3.16])**.** Suppose $f : X \to Y$ is a map of ∞-stacks on Cart that is an projective fibration. Then it is a fibration in the Čech model structure.

Definition C.5. If X, Y, Z are ∞-stacks on Cart, and the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ * \arrow[r, "z"'] & Z \end{tikzcd}$$

is a homotopy pullback square, where $* := \Delta^0$, then we say that the sequence of maps

$$X \to Y \to Z$$

is a **homotopy fiber sequence**, and we call X the **homotopy fiber** of f at z, which we sometimes denote by $\mathrm{hofib}(f)$.

Lemma C.6. Let $X \to Y \to Z$ be a homotopy fiber sequence of pointed ∞-stacks on Cart, and where the morphisms preserve the points. Then the resulting sequence

$$\check{H}^0_\infty(W, X) \xrightarrow{f} \check{H}^0_\infty(W, Y) \xrightarrow{g} \check{H}^0_\infty(W, Z),$$

is exact.[21]

Proof. This follows from the fact that $\mathbb{R}\check{\mathbb{H}}(-,-)$ preserves homotopy pullbacks in its second factor, so a homotopy fiber sequence of ∞-stacks produces a homotopy fiber sequence of spaces

$$\mathbb{R}\check{\mathbb{H}}(W, X) \to \mathbb{R}\check{\mathbb{H}}(W, Y) \to \mathbb{R}\check{\mathbb{H}}(W, Z)$$

and the long exact sequence of homotopy groups gives exactness for π_0. $\square$

Proposition C.7. Suppose that we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd} \tag{121}$$

of presheaves of chain complexes over Cart such that

$$\begin{tikzcd} \mathsf{DK}A \arrow[r, "\mathsf{DK}f"] \arrow[d, "\mathsf{DK}g"'] & \mathsf{DK}B \arrow[d, "\mathsf{DK}h"] \\ \mathsf{DK}C \arrow[r, "\mathsf{DK}h"'] & \mathsf{DK}D \end{tikzcd} \tag{122}$$

is a commutative diagram of ∞-stacks. If (121) is a homotopy pullback square in the projective model structure on ChPre(Cart), then (122) is a homotopy pullback square in $\check{\mathbb{H}}$.

Proof. If (121) is a homotopy pullback diagram, then A is weak equivalent to the actual pullback $C \times^h_D B$ of Lemma C.3. Since both of these presheaves of chain complexes are projective fibrant, and DK is right Quillen, then $\mathsf{DK}A$ is weak equivalent to $\mathsf{DK}(C \times^h_D B) \cong \mathsf{DK}C \times^h_{\mathsf{DK}D} \mathsf{DK}B$. Therefore $\mathsf{DK}A$ is a homotopy pullback of (122) in $\mathbb{H}$. Then by Proposition 3.5, it is a homotopy pullback in $\check{\mathbb{H}}$. $\square$

Now that we have all the technical tools we need, we restate Theorem 7.1 for the convenience of the reader.

[21] Exact in the sense that each set $\check{H}^0_\infty(W, A)$ is pointed by the constant map $*$ to the point of A, and the image of f is equal to the set of $x \in \check{H}^0_\infty(W, Y)$ such that $g(x) = *$, which we call the kernel of g.

Theorem 7.1. For every $k \geq 1$, there exists a commutative diagram of ∞-stacks of the following form

$$\begin{tikzcd}
* \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \mathbf{B}^k\mathbb{R}^\delta \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & * \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & * \arrow[d] \\
* \arrow[r] & \mathbf{B}^k_\nabla\mathbb{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "4"] & \Omega^{k+1}_{\mathrm{cl}} \arrow[r] \arrow[d] \arrow[dr, phantom, "5"] & \Omega^{k+1} \arrow[d] \\
 & \mathbf{B}^k\mathbb{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "6"] & \mathbf{B}^k\Omega^1_{\mathrm{cl}} \arrow[r] \arrow[d] \arrow[dr, phantom, "7"] & \Omega^{1\leq\bullet\leq k+1} \arrow[d] \\
 & * \arrow[r] & \mathbf{B}^{k+1}\mathbb{R}^\delta \arrow[r] & \mathbf{B}^{k+1}_\nabla\mathbb{R}
\end{tikzcd} \tag{123}$$

furthermore every commutative square in this diagram is a homotopy pullback square in $\check{\mathbb{H}}$.

Lemma C.8. The pasted square [4|5], given as follows

$$\begin{tikzcd}
\mathbf{B}^k_\nabla\mathbb{R} \arrow[r] \arrow[d] & \Omega^{k+1} \arrow[d] \\
\mathbf{B}^k\mathbb{R} \arrow[r] & \Omega^{1\leq\bullet\leq k+1}
\end{tikzcd} \tag{124}$$

is a homotopy pullback square in $\check{\mathbb{H}}$.

Proof. Let us analyze this part of the diagram as presheaves of chain complexes.

$$\begin{tikzcd}
{[\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k]} \arrow[r] \arrow[d] & {[0 \to 0 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}]} \arrow[r] \arrow[d] & {[0 \to 0 \to 0 \to \cdots \to \Omega^{k+1}]} \arrow[d] \\
{[\mathbb{R} \to 0 \to 0 \to \cdots \to 0]} \arrow[r] & {[\Omega^1 \to \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}]} \arrow[r] & {[\Omega^1 \to \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}]}
\end{tikzcd} \tag{125}$$

where the upper horizontal left hand map is 0 except in degree 0 where it applies the differential d. The lower horizontal left hand map is d in degree $k+1$ and 0 elsewhere. The rest of the maps are either degreewise inclusions or identity maps.

Let us show that the outer rectangle is a homotopy pullback diagram. Note that neither the bottom map nor the right hand map is objectwise surjective in positive degree, namely they are not fibrations in ChPre(Cart). However we can use Lemma C.3 to compute the homotopy pullback of $\mathbf{B}^k\mathbb{R} \to \Omega^{1\leq\bullet\leq k+1} \leftarrow \Omega^{k+1}$. Namely it is given as the actual (objectwise) pullback of the diagram

$$\begin{tikzcd}
 & (\Omega^{1\leq\bullet\leq k+1})^{\Delta^1} \arrow[d] \\
\mathbf{B}^k\mathbb{R} \oplus \Omega^{k+1} \arrow[r] & \Omega^{1\leq\bullet\leq k+1} \oplus \Omega^{1\leq\bullet\leq k+1}
\end{tikzcd} \tag{126}$$

Now $(\Omega^{1\leq\bullet\leq k+1})^{\Delta^1}$ is given by the presheaf of chain complexes

$$[\Omega^1 \oplus \Omega^1 \to \Omega^2 \oplus \Omega^2 \oplus \Omega^1 \to \cdots \to \Omega^k \oplus \Omega^k \oplus \Omega^{k-1} \to \Omega^{k+1} \oplus \Omega^k]$$

which projects to $\Omega^{1\leq\bullet\leq k+1} \oplus \Omega^{1\leq\bullet\leq k+1}$. From this we obtain the following diagram the following diagram of presheaves of chain complexes

$$\begin{tikzcd}
{[\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k]} \arrow[r] \arrow[d] & {[\Omega^1 \oplus \Omega^1 \to \Omega^2 \oplus \Omega^2 \oplus \Omega^1 \to \cdots \to \Omega^k \oplus \Omega^k \oplus \Omega^{k-1} \to \Omega^{k+1} \oplus \Omega^k]} \arrow[d, "\pi"] \\
{[\mathbb{R} \to 0 \to \cdots \to 0 \to \Omega^{k+1}]} \arrow[r] & {[\Omega^1 \oplus \Omega^1 \to \Omega^2 \oplus \Omega^2 \to \cdots \to \Omega^k \oplus \Omega^k \to \Omega^{k+1} \oplus \Omega^{k+1}]}
\end{tikzcd} \tag{127}$$

and this is an actual pullback square. To see this, note that pullbacks of chain complexes are computed degreewise. For degrees $k > 0$, it is clearly a pullback. In degree 0 we are trying to show that Ω^k is isomorphic to the pullback of

$$\begin{tikzcd} & \Omega^{k+1} \oplus \Omega^k \arrow[d, "\pi_0"] \\ 0 \oplus \Omega^{k+1} \arrow[r, "0\oplus 1_{\Omega^{k+1}}"'] & \Omega^{k+1} \oplus \Omega^{k+1} \end{tikzcd} \tag{128}$$

but from (118), we know that $\pi_0(x,z) = (x, x+dz)$. For every cartesian space U, the pullback is the set of triples $(w,x,z) \in \Omega^{k+1}(U) \oplus \Omega^{k+1}(U) \oplus \Omega^k(U)$ such that $x = 0$ and $w = dz$. This set is of course in bijection with $\Omega^k(U)$. Thus [4|5] is a homotopy pullback square. ☐

Lemma C.9. The square [5], given by

$$\begin{tikzcd} \Omega^{k+1}_{\mathrm{cl}} \arrow[r] \arrow[d] & \Omega^{k+1} \arrow[d] \\ \mathbf{B}^k\Omega^1_{\mathrm{cl}} \arrow[r] & \Omega^{1\leq\bullet\leq k+1} \end{tikzcd} \tag{129}$$

is a homotopy pullback square.

Proof. Consider the commutative diagram of presheaves of chain complexes

$$\begin{tikzcd} {[0 \to 0 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}]} \arrow[r] \arrow[d] & {[0 \to 0 \to \cdots \to \Omega^{k+1}]} \arrow[d] \\ {[\Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}]} \arrow[r] & {[\Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}]} \end{tikzcd} \tag{130}$$

Now the above diagram is an actual pullback, and the bottom map is objectwise a surjection in positive degrees, thus it is a fibration of presheaves of chain complexes, and therefore by Lemma C.1 the diagram (130) is a homotopy pullback. ☐

Corollary C.10. The square [4] is a homotopy pullback square.

Proof. By Lemma C.8, [4|5] is a homotopy pullback square. By Lemma C.9, [5] is a homotopy pullback square. Thus by Lemma C.2, [4] is a homotopy pullback square. ☐

Lemma C.11. The square [6]

$$\begin{tikzcd} \mathbf{B}^k\mathbb{R} \arrow[r] \arrow[d] & \mathbf{B}^k\Omega^1_{\mathrm{cl}} \arrow[d] \\ * \arrow[r] & \mathbf{B}^{k+1}\mathbb{R}^\delta \end{tikzcd} \tag{131}$$

is a homotopy pullback square.

Proof. This proof is very similar as the proof of Lemma C.8. We take the actual pullback of the diagram

$$\begin{tikzcd} & (\mathbf{B}^{k+1}\mathbb{R}^\delta)^{\Delta^1} \arrow[d] \\ 0 \oplus \mathbf{B}^k\Omega^1_{\mathrm{cl}} \arrow[r] & (\mathbf{B}^{k+1}\mathbb{R}^\delta \oplus \mathbf{B}^{k+1}\mathbb{R}^\delta) \end{tikzcd} \tag{132}$$

which by similar reasoning to the paragraph after (128) is precisely

(133)
$$\begin{tikzcd} {[0 \to \Omega^1 \oplus \mathbb{R} \xrightarrow{d_1} \Omega^2 \oplus \Omega^1 \xrightarrow{d_2} \dots \xrightarrow{d_{k-2}} \Omega^{k-1} \oplus \Omega^{k-2} \xrightarrow{\alpha} \Omega^{k-1}]} \arrow[r] \arrow[d] & {[\mathbb{R} \oplus \mathbb{R} \to \Omega^1 \oplus \Omega^1 \oplus \mathbb{R} \to \cdots \to \Omega^k_{\mathrm{cl}} \oplus \Omega^{k-1}]} \arrow[d, "\pi"] \\ {[0 \to \Omega^1 \to \cdots \to \Omega^k_{\mathrm{cl}}]} \arrow[r] & {[\mathbb{R} \oplus \mathbb{R} \to \Omega^1 \oplus \Omega^1 \to \cdots \to \Omega^k_{\mathrm{cl}} \oplus \Omega^k_{\mathrm{cl}}]} \end{tikzcd}$$

where $d_i(a,b) = (da, db - (-1)^i a)$ for $1 \leq i \leq k-2$, and $\alpha(a,b) = db + a$. Now there is an obvious map

(134)
$$[0 \to \mathbb{R} \to \cdots \to 0] \to [0 \to \Omega^1 \oplus \mathbb{R} \xrightarrow{d_1} \Omega^2 \oplus \Omega^1 \xrightarrow{d_2} \dots \xrightarrow{d_{k-2}} \Omega^{k-1} \oplus \Omega^{k-2} \xrightarrow{\alpha} \Omega^{k-1}]$$

that is an isomorphism on cohomology. Indeed α is surjective, and the kernel of $d_i : \Omega^i \oplus \Omega^{i-1} \to \Omega^{i+1} \oplus \Omega^i$ is the set of pairs (a,b) where $a = (-1)^i db$, and these are all in the image of d_{i-1}. □

Lemma C.12. The square [7]

(135)
$$\begin{tikzcd} \mathbf{B}^k \Omega^1_{\mathrm{cl}} \arrow[r] \arrow[d] & \Omega^{1 \leq \bullet \leq k+1} \arrow[d] \\ \mathbf{B}^{k+1} \mathbb{R}^\delta \arrow[r] & \mathbf{B}^{k+1}_\nabla \mathbb{R} \end{tikzcd}$$

is a homotopy pullback square.

Proof. As presheaves of chain complexes we have

(136)
$$\begin{tikzcd} {[0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}]} \arrow[r] \arrow[d] & {[0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}]} \arrow[d] \\ {[\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}]} \arrow[r] & {[\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}]} \end{tikzcd}$$

which is an actual pullback, and the bottom horizontal map is a fibration, thus by Lemma C.1, [7] is a homotopy pullback. □

Lemma C.13. The pasted square $[\frac{2}{4}]$,

(137)
$$\begin{tikzcd} \mathbf{B}^k \mathbb{R}^\delta \arrow[r] \arrow[d] & * \arrow[d] \\ \mathbf{B}^k \mathbb{R} \arrow[r] & \mathbf{B}^k \Omega^1_{\mathrm{cl}} \end{tikzcd}$$

is a homotopy pullback square.

Proof. We use the same proof technique as in Lemma C.8, namely we will compute the actual pullback of the diagram

(138)
$$\begin{tikzcd} & (\mathbf{B}^k \Omega^1_{\mathrm{cl}})^{\Delta^1} \arrow[d] \\ \mathbf{B}^k \mathbb{R} \oplus 0 \arrow[r] & \mathbf{B}^k \Omega^1_{\mathrm{cl}} \oplus \mathbf{B}^k \Omega^1_{\mathrm{cl}} \end{tikzcd}$$

The actual pullback we obtain is given by

(139)
$$\begin{tikzcd} {[\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k_{\mathrm{cl}}]} \arrow[r] \arrow[d] & {[\Omega^1 \oplus \Omega^1 \to \Omega^2 \oplus \Omega^2 \oplus \Omega^1 \to \cdots \to \Omega^k_{\mathrm{cl}} \oplus \Omega^k_{\mathrm{cl}} \oplus \Omega^{k-1} \to 0 \oplus \Omega^k_{\mathrm{cl}}]} \arrow[d, "\pi"] \\ {[\mathbb{R} \to 0 \to \cdots \to 0 \to 0]} \arrow[r] & {[\Omega^1 \oplus \Omega^1 \to \Omega^2 \oplus \Omega^2 \to \cdots \to \Omega^k_{\mathrm{cl}} \oplus \Omega^k_{\mathrm{cl}} \to 0]} \end{tikzcd}$$

which is similar to the computation (127). Thus $[\frac{2}{4}]$ is a homotopy pullback square. □

Corollary C.14. The square [2] is a homotopy pullback square.

Proof. By Corollary C.10, Lemma C.13 and Lemma C.2. □

Lemma C.15. The square [3] is a homotopy pullback square.

Proof. As a diagram of presheaves of chain complexes

$$\begin{tikzcd} {[0 \to \Omega^1 \to \cdots \to \Omega^k]} \arrow[r] \arrow[d] & {[\mathbb{R} \to \Omega^1 \to \cdots \to \Omega^k]} \arrow[d] \\ {[0 \to 0 \to \cdots \to 0]} \arrow[r] & {[\mathbb{R} \to 0 \to \cdots \to 0]} \end{tikzcd} \tag{140}$$

it is an actual pullback, and the right hand map is a fibration. □

Lemma C.16. The square [1] is a homotopy pullback square.

Proof. As a diagram of presheaves of chain complexes

$$\begin{tikzcd} {[0 \to \cdots \to 0]} \arrow[r] \arrow[d] & {[\mathbb{R} \to \cdots \to \Omega^k_{\mathrm{cl}}]} \arrow[d] \\ {[0 \to \cdots \to 0]} \arrow[r] & {[\mathbb{R} \to \cdots \to \Omega^k]} \end{tikzcd} \tag{141}$$

it is an actual pullback, and the right hand map is a fibration. □

Thus we have proven Theorem 7.1.

References

[ADH21] Araminta Amabel, Arun Debray, and Peter J. Haine. *Differential Cohomology: Categories, Characteristic Classes, and Connections*. 2021. arXiv: 2109.12250 [math.AT].

[Ahm23] Alireza Ahmadi. *Diffeological Čech cohomology*. 2023. arXiv: 2303.03251 [math.DG].

[Ara23] Kensuke Arakawa. *Homotopy Limits and Homotopy Colimits of Chain Complexes*. 2023. arXiv: 2310.00201 [math.AT].

[BBP22] Daniel Berwick-Evans, Pedro Boavida de Brito, and Dmitri Pavlov. *Classifying spaces of infinity-sheaves*. 2022. arXiv: 1912.10544 [math.AT].

[BH11] John Baez and Alexander Hoffnung. "Convenient categories of smooth spaces". *Transactions of the American Mathematical Society* 363.11 (2011), pp. 5789–5825.

[BK72] Aldridge Knight Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and localizations*. Vol. 304. Springer Science & Business Media, 1972.

[Bor48] Karol Borsuk. "On the imbedding of systems of compacta in simplicial complexes". *Fundamenta Mathematicae* 35.1 (1948), pp. 217–234.

[BT+82] Raoul Bott, Loring W Tu, et al. *Differential forms in algebraic topology*. Vol. 82. Springer, 1982.

[Bun21a] Severin Bunk. "Gerbes in geometry, field theory, and quantisation". *Complex Manifolds* 8.1 (2021), pp. 150–182.

[Bun21b] Severin Bunk. *Sheaves of Higher Categories on Generalised Spaces*. 2021. arXiv: 2003.00592 [math.AT].

[Bun22] Severin Bunk. "The ℝ-local homotopy theory of smooth spaces". *Journal of Homotopy and Related Structures* (2022), pp. 1–58.

[Car15] David Carchedi. *On The Homotopy Type of Higher Orbifolds and Haefliger Classifying Spaces*. 2015. arXiv: 1504.02394 [math.AT].

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Cambridge University Press, 2019.

[Clo23] Adrian Clough. *The homotopy theory of differentiable sheaves*. 2023. arXiv: 2309.01757 [math.AT].

[CW14] J Daniel Christensen and Enxin Wu. "The homotopy theory of diffeological spaces". *New York J. Math* 20 (2014), pp. 1269–1303.

[CW15] J. Daniel Christensen and Enxin Wu. *Tangent spaces and tangent bundles for diffeological spaces*. 2015. arXiv: 1411.5425 [math.DG].

[deR31] Georges deRham. "Sur l'analysis situs des variétés à n dimensions". *Journal de mathématiques pures et appliquées* 10 (1931), pp. 115–200.

[DHI04] Daniel Dugger, Sharon Hollander, and Daniel C Isaksen. "Hypercovers and simplicial presheaves". *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 136. 1. Cambridge University Press. 2004, pp. 9–51.

[Dug01] Daniel Dugger. "Universal homotopy theories". *Advances in Mathematics* 164.1 (2001), pp. 144–176.

[FH13] Daniel Freed and Michael Hopkins. "Chern–Weil forms and abstract homotopy theory". *Bulletin of the American Mathematical Society* 50.3 (2013), pp. 431–468.

[FSS+12] Domenico Fiorenza, Urs Schreiber, Jim Stasheff, et al. "Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction". *Adv. Theor. Math. Phys* 16.1 (2012), pp. 149–250.

[GJ12] Paul G Goerss and John F Jardine. *Simplicial Homotopy Theory*. Vol. 174. Birkhäuser, 2012.

[GQ22] Jean Gallier and Jocelyn Quaintance. *Homology, Cohomology, and Sheaf Cohomology for Algebraic Topology, Algebraic Geometry, and Differential Geometry*. World Scientific, 2022.

[GS06] Paul G. Goerss and Kristen Schemmerhorn. "Model Categories and Simplicial Methods" (2006). arXiv: math/0609537 [math.AT].

[Hir09] Philip S Hirschhorn. *Model categories and their localizations*. 99. American Mathematical Soc., 2009.

[Hov07] Mark Hovey. *Model categories*. 63. American Mathematical Soc., 2007.

[Igl13] Patrick Iglesias-Zemmour. *Diffeology*. Vol. 185. American Mathematical Soc., 2013.

[Igl20] Patrick Iglesias-Zemmour. *The Irrational Toruses*. http://math.huji.ac.il/~piz/documents/ShD-lect-TIT.pdf. Accessed: 2023-10-23. 2020.

[Igl23] Patrick Iglesias-Zemmour. "Čech–De Rham bicomplex in diffeology". *Israel Journal of Mathematics* (2023), pp. 1–38.

[Igl88] Patrick Iglesias-Zemmour. *Une cohomologie de Cech pour les espaces differentiables et sa relation a la cohomologie de De Rham*. https://math.huji.ac.il/~piz/documents/UCDCPLEDESRALCDDR.pdf. Accessed: 2023-10-12. 1988.

[Jar] John F. Jardine. *Homotopy Colimits*. https://www.uwo.ca/math/faculty/jardine/courses/homalg/homalg-lecture009.pdf. Accessed: 2023-10-12.

[Jar03] John F Jardine. "Presheaves of chain complexes". *K-theory* 30.4 (2003), pp. 365–420.

[Jaz21] David Jaz Myers. *Modal Fracture of Higher Groups*. June 2021. arXiv: math.CT/2106.15390.

[KWW21] Derek Krepski, Jordan Watts, and Seth Wolbert. *Sheaves, principal bundles, and Čech cohomology for diffeological spaces*. 2021. arXiv: 2111.01032 [math.DG].

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Lur17] Jacob Lurie. *Higher algebra*. 2017.

[Min22] Emilio Minichiello. "Diffeological Principal Bundles and Principal Infinity Bundles" (2022). arXiv: 2202.11023 [math.DG].

[MM12] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.

[Mur96] Michael K Murray. "Bundle gerbes". *Journal of the London Mathematical Society* 54.2 (1996), pp. 403–416.

[MW17] Jean-Pierre Magnot and Jordan Watts. "The diffeology of Milnor's classifying space". *Topology and its Applications* 232 (2017), pp. 189–213.

[Opa21] Michael Mayowa Opadotun. "Simplicial Enrichment of Chain Complexes". PhD thesis. The University of Regina (Canada), 2021.

[Pav22a] Dmitri Pavlov. "Numerable open covers and representability of topological stacks". *Topology and its Applications* 318 (2022), p. 108203.

[Pav22b] Dmitri Pavlov. *Projective model structures on diffeological spaces and smooth sets and the smooth Oka principle*. 2022. arXiv: 2210.12845 [math.AT].

[Rez10] Charles Rezk. *Toposes and homotopy toposes*. 2010. URL: https://ncatlab.org/nlab/files/Rezk_HomotopyToposes.pdf.

[Ric] Jeremy Rickard. *Sign convention for total complex*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/3147981.

[Rie14] Emily Riehl. *Categorical homotopy theory*. Vol. 24. Cambridge University Press, 2014.

[Sch13] Urs Schreiber. *Differential cohomology in a cohesive infinity-topos*. 2013. arXiv: 1310.7930 [math-ph].

[SS03] Stefan Schwede and Brooke Shipley. "Equivalences of monoidal model categories". *Algebraic & Geometric Topology* 3.1 (2003), pp. 287–334.

[SS21] Hisham Sati and Urs Schreiber. *Equivariant principal infinity-bundles*. 2021. arXiv: 2112.13654.

[Ste04] Daniel Stevenson. "Bundle 2-gerbes". *Proceedings of the London Mathematical Society* 88.2 (2004), pp. 405–435.

[Wal12] Konrad Waldorf. "Transgression to loop Spaces and its inverse, I: Diffeogical Bundles and Fusion Maps". *Cahiers de topologie et géométrie différentielle catégoriques* 53.3 (2012), pp. 162–210.

[Wei95] Charles A Weibel. *An introduction to homological algebra*. 38. Cambridge university press, 1995.