A MATHEMATICAL MODEL OF PACKAGE MANAGEMENT SYSTEMS - FROM GENERAL EVENT STRUCTURES TO ANTIMATROIDS

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Abstract. This paper brings mathematical tools to bear on the study of package dependencies in software systems. We introduce structures known as Dependency Structures with Choice (DSC) that provide a mathematical account of such dependencies, inspired by the definition of general event structures in the study of concurrency. We equip DSCs with a particular notion of morphism and show that the category of DSCs is isomorphic to the category of antimatroids. We study the exactness properties of these equivalent categories, and show that they are finitely complete, have finite coproducts but not all coequalizers.

Further, we show construct a functor from a category of DSCs equipped with a certain subclass of morphisms to the opposite of the category of finite distributive lattices, making use of a simple finite characterization of the Bruns-Lakser completion, and finally, we introduce a formal account of versions of packages and introduce a mathematical account of package version-bound policies.

Contents

1. Introduction

Package repositories and package management systems are pervasive in modern software. Such repositories can consist of binary packages in Linux distributions (such as Debian or Arch), or of source packages for use in the course of developing software and managing the libraries on which it depends (such as the npm package repository for the JavaScript language or the Hackage package repository for the Haskell language).

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This paper works towards the development of a basic toolkit for considering package dependencies in software systems. Specifically we develop a mathematical model of package management systems which we call **Dependency Structures with Choice** or **DSC**s for short. A preDSC consists of a finite set E, whose elements we think of as packages, and a function $\text{dep} : E \to P(P(E))$, where $P(E)$ denotes the power set of E. We think of this function as the dependency structure on E. To every package e we can associate a collection of "possible dependency sets" $D^e \in \text{dep}(e)$. This models how packages can be run using various different sets of other packages. The inspiration for this mathematical model comes from general event structures, which were introduced for the purpose of studying concurrent computation [NPW79]. If we also require that dep is "irredundant," namely that if $X, Y \in \text{dep}(e)$ and $X \subseteq Y$, then $X = Y$, then irredundant preDSCs are actually equivalent to general event structures that have no conflict. This is the content of Proposition 2.6.

While irredundant preDSCs are equivalent to conflict free general event structures, we may demand more properties of preDSCs that are natural for package management systems. By demanding these properties, we come to the notion of a DSC. To every DSC, which we denote by (E, dep), we can associate a poset $\mathsf{rdp}(E)$, known as its **reachable dependency poset**, whose elements are those subsets of E that are "reachable." One can think of a subset $A \subseteq E$ of packages as being reachable if the elements of A can be ordered into a sequence of package installations such that each new package installed depends only on already installed packages. It turns out that reachable dependency posets of DSCs inherit more structure, namely they are finite join-semilattices, which implies that they are finite lattices. This structure gives us an order-theoretic way of describing and analyzing the dependency structure inherent to a DSC.

We define a notion of morphism for DSCs in such a way that rdp becomes a functor from the category of DSCs to the category of finite join-semilattices. If E is a DSC, then $\mathsf{rdp}(E)$ will always be in a certain class of lattices, namely those that are diamond-free semimodular, which is defined in Definition 3.20. These diamond-free semimodular lattices are intimately related with antimatroids. This combination of facts led the authors to discover a correspondence between DSCs and antimatroids.

Antimatroids have been discovered and rediscovered for many purposes in lattice theory, graph theory, and combinatorics since the 1940s [Mon85]. An antimatroid consists of a set E and a collection of subsets $\mathcal{F}$, whose elements are called feasible sets, satisfying the axioms of Definition 3.1. Antimatroids have several applications in computer science, such as in [GY94] and [MR16]. We provide another application by showing that if (E, dep) is a DSC, then the pair $(E, \mathsf{rdp}(E))$ is an antimatroid. This is the content of Proposition 3.9. The literature on antimatroids does not provide a standard definition of what a morphism of antimatroids should be. We propose that morphisms of antimatroids should be those functions that pull feasible sets back to feasible sets. This natural choice defines a category of antimatroids. One of the main results of this paper is Theorem 3.15, which states that there is an isomorphism between the category of DSCs and the category of antimatroids. Combining this theorem with a result of Czedli [Czé14] then shows that any diamond-free semimodular lattice is isomorphic to the reachable dependency poset of a DSC.

We prove exactness properties for the category of DSCs/antimatroids, which might be of independent interest. In particular we prove that this category has all finite limits, but not all finite colimits. In particular, while the category of DSCs/antimatroids has finite coproducts, not all coequalizers exist. This can be seen in contrast to [HP18], which characterizes exactness properties of the category of matroids.

Given a DSC (E, dep), its reachable dependency poset $\mathsf{rdp}(E)$ will in general not be a distributive lattice. One can complete a lattice using the Bruns-Lakser completion or injective envelope [BL70]. When applied to the reachable dependency poset of a DSC, the Bruns-Lakser completion defines a Merkle tree, which gives a complete description of the execution traces of the DSC. We give an easily computed description of the Bruns-Lakser completion for finite lattices in Proposition 6.16 (implicit in the work of [GG13]) which is of independent interest.

Lastly, we axiomatize what it means for a package to be an updated version of another package in a DSC, characterizing it as an idempotent monad on $\mathsf{rdp}(E)$. We extend this to an idempotent monad on the Bruns-Lakser completion of $\mathsf{rdp}(E)$ as well.

The paper is organized as follows. In Section 2 we define DSCs, the reachable dependency poset construction, morphisms of DSCs, and show that with these definitions, rdp defines a contravariant functor from the category of DSCs to the category of finite join-semilattices. We show that by restricting to a smaller class of morphisms, rdp extends to a functor to the category of finite lattices. In Section 3 we prove that the categories of DSCs and the category of antimatroids are isomorphic. From this we prove that a lattice is diamond-free semimodular if and only if it can be written as the reachable dependency poset of a DSC. In Section 4, we study a subclass of DSCs where every element has a unique possible dependency set and show that it is equivalent to the opposite of the category of finite posets. In Section 5 we study the exactness properties of the category of DSCs/antimatroids. In Section 6, we study the Bruns-Lakser completion of reachable dependency posets, showing that they produce Merkle trees. Finally in Section 7 we define version parametrization for DSCs.

1.1. Relation to Other Work.

In some ways, this paper is an update to and expansion of [BP20]. The paper [Baz21] also studies DSCs, but with a different motivation and focus. This paper can be read completely independently of [BP20] and [Baz21].

1.2. Notation.

In what follows we use the following notation.

1. If (E, dep) is a preDSC, and $e \in E$, we often let D^e denote a possible dependency set of e,
2. Given a set function $f : A \to B$, let $f_* : P(A) \to P(B)$ denote the image morphism, namely $f_*(S) = \{f(s) \in B : s \in S\}$, and let $f^* : P(B) \to P(A)$ denote the preimage morphism, namely $f^*(T) = \{a \in A : f(a) \in T\}$.
3. We will often denote singleton sets $\{a\}$ without their brackets, especially in expressions like $A \setminus \{a\}$ or $A \cup \{a\}$, we will write this as $A \setminus a$ or $A \cup a$,
4. We often denote categories with the type face C, and the set of its objects by C_0.

2. Dependency Structures with Choice

2.1. General Event Structures.

In this section we recall general event structures, first defined in [NPW79], and consider their restriction to general event structures without conflict. We take this as the starting point for our definition of pre-dependency structures with choice. We then motivate further axioms we require in order to define a dependency structure with choice.

Definition 2.1. A **general event structure** is a triple $(E, \mathrm{Con}, \dashv)$ where:

(1) E is a finite set, whose elements we call **events**,
(2) $\mathrm{Con} \subseteq P(E)$, is a collection of subsets called the **consistency predicate** such that if $Y \in \mathrm{Con}$ and $X \subseteq Y$, then $X \in \mathrm{Con}$, and
(3) $\dashv \subseteq \mathrm{Con} \times E$ is a relation called the **enabling relation**, which satisfies the following property: if $X \dashv e$, $X \subseteq Y$, and $Y \in \mathrm{Con}$, then $Y \dashv e$.

In this paper, we wish to restrict attention to only those general event structures which are conflict-free.

Definition 2.2. A **conflict-free general event structure** is a general event structure $(E, \mathrm{Con}, \dashv)$ where $\mathrm{Con} = P(E)$.

Now the enabling relation $\vdash \subseteq P(E) \times E$ is equivalent to a function $\mathrm{en} : E \to P(P(E))$. Indeed, given a relation $\vdash \subseteq P(E) \times E$, let $\mathrm{en}(e) = \{X \subseteq E : X \vdash e\}$. Conversely if $\mathrm{en} : E \to P(P(E))$ is a function, then let $X \vdash e$ if $X \in \mathrm{en}(e)$.

We can therefore say that a conflict-free general event structure consists of a set E equipped with a function $\mathrm{en} : E \to P(P(E))$ such that if $X \in \mathrm{en}(e)$ and $X \subseteq Y$, then $Y \in \mathrm{en}(e)$. This form better lends itself to modelling package management systems.

Indeed if we think of an event $e \in E$ of a conflict-free general event structure as a package, then we can think of $\mathrm{en}(e)$ as those sets of packages which e might depend on. For instance, suppose $E = \{a, b, c, d\}$, and $\mathrm{en}(a) = \{\{b, c\}, \{d\}\}$. This would be interpreted as a logical formula:

$$\mathrm{en}(a) = (b \wedge c) \vee d$$

where $\wedge$ means "and," and $\vee$ means "or." Thus we would read this as "a depends on either the set of packages $\{b, c\}$ or the set of packages $\{d\}$."

This is an important structure for a package management system to have. For example, suppose we have packages a, b, c_1, c_2 where c_i are various versions of some package c. For instance we might think of c_1 as `c-0.1` and c_2 as `c-0.2`. Then a might be able to depend on either version of c, so $\mathrm{dep}(a)$ might be $(b \wedge c_1) \vee (b \wedge c_2)$, so that a depends on either b and c_1 or b and c_2.

Notice that if we can run a package e with a set of packages X, then we should not care that we can then run e with $X \cup Y$ for some set Y (recall we are not considering conflicts in this model). We should only care about those minimal possible set of other packages with which we can run e.

Definition 2.3. Let $(E, \mathrm{Con}, \dashv)$ be a general event structure. If $X \subseteq E$, and $X \dashv e$, then we say that X is a **minimal enabling** of e and write $X \dashv_0 e$ if for every $Y \subseteq X$ such that $Y \dashv e$, we have that $X = Y$. Let $\mathrm{en}_0(e) = \{X \subseteq E : X \dashv_0 e\}$. (Adapted from [Mon+18, Definition 2.5])

Now let (E, en) be a conflict-free event structure. If $e \in E$ we can consider the function $\mathrm{en}_0 : E \to P(P(E))$ given by the minimal enabling sets.

Definition 2.4. A **pre-dependency structure with choice** or **preDSC** is a pair

$$\left(E, E \xrightarrow{\text{dep}} P(P(E))\right),$$

where E is a finite set. We say that the function dep is **irredundant** if whenever $X, Y \in \text{dep}(e)$ and $X \subseteq Y$, then $X = Y$. We call such a pair (E, dep) an **irredundant preDSC**.

Example 2.5. The concrete data of a package repository is syntactically very close to the data in a preDSC, so this representation is natural to begin with from the perspective of software engineering. For example, below is a `package.json` file used to describe JavaScript packages in the npm package repository:

```
{
  "name": "leftpad",
  "version": "5.9.2",
  "description": "Provides left padding",
  "main": "index.js",
  "license": "MIT",
  "dependencies": {
      "react": "2.4.0",
      "webpack": "0.1.3",
      "redis": "^4.3.0"
   }
}
```

The name and version properties serve to uniquely identify the package within a given repository. The description and license are intended, for the most part, for human consumption, and we need not consider them here. The "main" property is used to indicate the entry point of the package to the build system, but we are not concerned with the actual build process at this moment, so it need not be considered either. The central thing to understand is then the "dependencies" property. Here, it is given as a mapping of package names to package version specifications. So the `react` package is required to already be available at precisely version 2.4.0, and similarly for `webpack`. On the other hand, a caret precedes the version specification for `redis`. In the syntax of these files, this indicates that any version of the package with major version 4 and minor version ≥ 3 is acceptable, i.e. it specifies not just a single version, but a range. Supposing we knew for a fact that the matching versions were 4.3.0, 4.4.0 and 4.5.0, then the meaning of this dependency would be the disjunctive clause: `redis-4.3.0` $\vee$ `redis-4.4.0` $\vee$ `redis-4.5.0`. Further, since each package gives a disjunction of versions, but the dependency field as a whole specifies a conjunction of packages, then the entirety of the key metadata for a package may be regarded as a single formula:

leftpad-5.9.2 $\to$ react-2.4.0 $\wedge$ webpack-0.1.3 $\wedge$ (redis-4.3.0 $\vee$ redis-4.4.0 $\vee$ redis-4.5.0).

Let us now compare conflict-free event structures with irredundant preDSCs. Let CFES_0 denote the set[1] of conflict-free event structures, and let $\mathsf{ipreDSC}_0$ denote the set of irredundant preDSCs. There is a function $\varphi : \mathsf{CFES}_0 \to \mathsf{ipreDSC}_0$ defined by setting $\varphi(E, \mathrm{en}) = (E, \mathrm{en}_0)$. Conversely, given an irredundant preDSC (E', dep), let $\psi(E', \mathrm{dep})$ be the conflict-free event structure with $\mathrm{en}' : E' \to P(P(E'))$ defined by $\mathrm{en}'(e) = \{Y \subseteq E' : \exists X \subseteq Y, X \in \mathrm{dep}(e)\}$. It is not hard to see that these maps are inverse to each other. Thus we have shown the following.

Proposition 2.6. The maps φ, ψ above define a bijection between conflict-free general event structures and irredundant preDSCs,

$$\mathsf{CFES}_0 \cong \mathsf{ipreDSC}_0.$$

Now we wish to restrict to certain subclasses of preDSCs that better approximate real-life package management systems. Given a pre-DSC (E, dep), we call elements $e \in E$ **events** and subsets $X \subseteq E$ **event sets**. We call the elements $Y \in \mathrm{dep}(e)$ **possible dependency sets** or **depsets** for e.

Definition 2.7. Given a preDSC (E, dep), we call an event set $X \subseteq E$ **complete** if for every $e \in X$, there exists a depset $D^e \in \mathrm{dep}(e)$ such that $D^e \subseteq X$.

Example 2.8. If $E = \{a, b, c\}$ and

$$\begin{aligned} \mathrm{dep}(a) &= \{\{b, c\}\} \\ \mathrm{dep}(b) &= \{\{a\}, \{c\}\} \\ \mathrm{dep}(c) &= \{\varnothing\} \end{aligned}$$

then $X = \{b, c\}$ is a complete event set, but $\{a\}$ and $\{b\}$ are not.

Definition 2.9. We say a preDSC (E, dep) is a **dependency structure with choice** or a **DSC** if

- (D0) dep is an irredundant function,
- (D1) for every $e \in E$, $\mathrm{dep}(e)$ is nonempty,
- (D2) if $X \in \mathrm{dep}(e)$, then $e \notin X$, and
- (D3) for every $e \in E$, and for every depset $X \in \mathrm{dep}(e)$, X is complete.

Remark 2.10. Note that condition (D1) does not mean that e cannot depend on nothing, because $\{\varnothing\}$ is an element of $P(P(E))$.

Remark 2.11. Note that (D3) amounts to a form of transitive closure. In particular, if b is a dependency of a and c is a dependency of b then c will also be a dependency of a. (D2) which states that no event may depend on itself, when coupled with (D3), ensures a global cycle-freeness, no chain of dependencies may loop back on itself.

[1] this is not truly a set, but a class. We could consider a skeleton of the category of sets and then restrict to such sets equipped with the above structure to resolve this issue, however set-theoretic issues will be of no concern in this paper and we thus ignore the discrepancy.

2.2. **Reachable Dependency Posets.** In this section we show that we can associate to any DSC a poset, known as its reachable dependency poset, that we will show is actually a lattice. The reachable dependency poset gives an order structure to the reachable states that DSCs model.

Let (E, dep) be a preDSC. We will construct a relation rch on $P(E)$ as follows.

Definition 2.12. Let X, Y be event sets of E. We define a relation $\mathsf{rch} \subseteq P(E) \times P(E)$, short for reach, as follows. We write $X \,\mathsf{rch}\, Y$ if

(1) $X \subseteq Y$, and
(2) for every $y \in Y$, there exists a depset $D^y \in \mathrm{dep}(y)$ such that $D^y \subseteq X$.

Lemma 2.13. An event set $X \subseteq E$ is complete if and only if $U \,\mathsf{rch}\, X$ for some subset $U \subseteq X$.

Note that if E is a DSC, then rch is reflexive on every possible dependency set, but that does not make it a reflexive relation on all event sets. Also note that rch is anti-symmetric, i.e. if $X \,\mathsf{rch}\, Y$ and $Y \,\mathsf{rch}\, X$, then $X = Y$.

Definition 2.14. Given a DSC (E, dep), we say that an event set $X \subseteq E$ is **reachable** if there exists a finite sequence $X_0, X_1, \dots, X_n$ of event sets such that

$$\varnothing \,\mathsf{rch}\, X_0 \,\mathsf{rch}\, X_1 \,\mathsf{rch}\, \dots \,\mathsf{rch}\, X_n \,\mathsf{rch}\, X.$$

Remark 2.15. The intuition one might have for rch is that $X \,\mathsf{rch}\, Y$ if everything in Y can depend on things in X. With regards to package management, one can imagine that a set of packages is reachable if, starting from the empty state, a sequence of package installs can eventually produce that precise set of installed packages. We will show in Proposition 2.27 that an event set is reachable if and only if it is complete. Thus one can think of a reachable event set as a set of packages that can be installed together with no further dependencies.

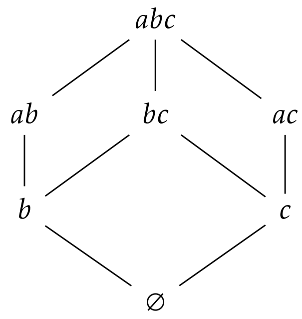
Lemma 2.16. If (E, dep) is a DSC, and $A \,\mathsf{rch}\, A'$, $B \,\mathsf{rch}\, B'$, then $A \cup B \,\mathsf{rch}\, A' \cup B'$.

Proof. If every event in A' can depend on events in A, and every event in B' can depend on events in B, then every event in $A' \cup B'$ can depend on events in A or events in B, thus it can depend on events in $A \cup B$. ☐

Definition 2.17. Given a preDSC (E, dep), let $\mathsf{rdp}(E)$ denote the subset of $P(E)$ consisting of the reachable event sets. This inherits a partial order from the partial order $\subseteq$ on $P(E)$. We call $(\mathsf{rdp}(E), \subseteq)$ the **reachable dependency poset** of E.

Example 2.18. Let $E = \{a, b, c\}$, and $\mathrm{dep}(a) = \{\{b\}, \{c\}\}$, namely a can depend on b or c, and $\mathrm{dep}(b) = \mathrm{dep}(c) = \varnothing$. Notice that this is a DSC. Consider its reachable dependency poset $\mathsf{rdp}(E)$. Recall the notion of a Hasse diagram, namely draw an event set X above Y if X covers Y. By this we mean that $Y \subseteq X$ and if $Y \subseteq Z \subseteq X$, then $Z = Y$ or $Z = X$. In our diagram, we write an event set like $\{a, b, c\}$ as abc.

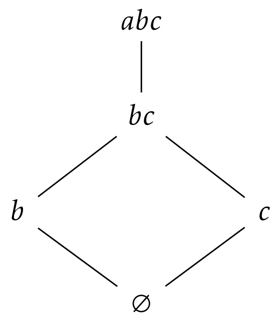
(1)



For shorthand we will refer to this example as $a.b \vee c$, to mean that a depends on b or c which in turn do not depend on anything.

Example 2.19. Let us compare Example 2.18 with a DSC $E = \{a, b, c\}$ where now $\mathrm{dep}(a) = \{\{b, c\}\}$, and $\mathrm{dep}(b) = \mathrm{dep}(c) = \emptyset$, namely a depends on b and c. In this case, the Hasse diagram of $\mathsf{rdp}(E)$ is

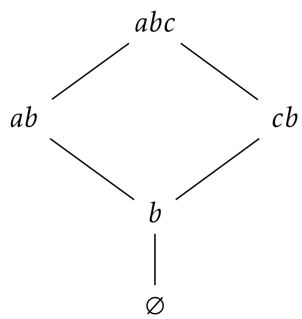
(2)



For shorthand we denote this DSC by $a.b \wedge c$.

Example 2.20. Consider $E = \{a, b, c\}$ and $\mathrm{dep}(a) = \{b\}, \mathrm{dep}(b) = \emptyset, \mathrm{dep}(c) = \{b\}$. This has $\mathsf{rdp}(E)$ given by the Hasse diagram:

(3)



For shorthand we will denote this DSC by $a.b, c.b$, to mean that a depends on b and c depends on b.

We now recall the following definitions, see [Bir40]. In this paper we will only consider finite posets.

Definition 2.21. Let $(P, \leq)$ be a poset, and $a, b \in P$.

- We say an element $z \in P$ is the **join** of a and b if it is the lowest upper bound or supremum of a and b, i.e. $z \geq a$, $z \geq b$ and if $w \geq a$, $w \geq b$, then $w \geq z$. If the join of a and b exists, then it is unique and we denote it by $a \vee b$. A poset with binary joins is called a **join-semilattice**. A morphism of join-semilattices is an order preserving map that preserves joins.
- For a subset $S \subseteq P$ of a join-semilattice, we let $\bigvee S$ denote the supremum of all the elements of S. If $S = P$, then we denote $\bigvee P = \top$, and call this the **top element** of P.
- Similarly we define the **meet** of a and b to be the greatest lower bound or infimum of a and b, and we denote it by $a \wedge b$. A poset with binary meets is called a **meet-semilattice**. A morphism of meet-semilattices is an order preserving map that preserves meets.
- For a subset $S \subseteq P$ of a meet-semilattice, let $\bigwedge S$ denote the infimum of all the elements of a finite subset S. If $S = P$, then we denote $\bigwedge S = \bot$ and call this the **bottom element** of P.
- A poset which is both a join-semilattice and a meet-semilattice is called a **lattice**. A morphism of lattices is an order preserving map that preserves both joins and meets.

Let Pos denote the category of posets with order preserving maps, and similarly denote $\mathsf{MSLatt}, \mathsf{JSLatt}, \mathsf{Latt}$ for the categories of meet-semilattices, join-semilattices, and lattices respectively.

Lemma 2.22. If (E, dep) is a DSC, and $X, Y \subseteq E$ are reachable event sets, then $X \cup Y$ is a reachable event set, and is the join of X and Y in $\mathsf{rdp}(E)$.

Proof. If X and Y are reachable, then there exist sequences $\{X_i\}$ and $\{Y_i\}$ such that

$$\varnothing \,\mathsf{rch}\, X_1 \,\mathsf{rch}\, X_2 \,\mathsf{rch} \ldots \mathsf{rch}\, X_n \,\mathsf{rch}\, X$$

$$\varnothing \,\mathsf{rch}\, Y_1 \,\mathsf{rch}\, Y_2 \,\mathsf{rch} \ldots \mathsf{rch}\, Y_m \,\mathsf{rch}\, Y$$

Suppose WLOG that $n \geq m$, then by Lemma 2.16 we have

$$\varnothing \,\mathsf{rch}\, X_1 \cup Y_1 \,\mathsf{rch} \ldots \mathsf{rch}\, X_m \cup Y_m \,\mathsf{rch}\, X_{m+1} \cup Y_m \,\mathsf{rch} \ldots \mathsf{rch}\, X_n \cup Y_m \,\mathsf{rch}\, X \cup Y,$$

thus $X \cup Y$ is reachable. It is easy to see that $X \cup Y$ must then be the join of X and Y in $\mathsf{rdp}(E)$. □

Corollary 2.23. Given a DSC (E, dep), its reachable dependency poset $\mathsf{rdp}(E)$ is a finite join-semilattice.

Remark 2.24. Notice that the intersection of two reachable event sets might not be reachable. Using the same DSC $a.b \vee c$ from Example 2.18 we see that while $\{a, b\}$ and $\{a, c\}$ are reachable, their intersection $\{a\}$ is not.

In order to better understand rdp, it would be useful to find another characterization of reachable event sets that is easier to check. Let E be a DSC and $A \subseteq E$. If $a, b \in A$, then we write $a \lhd_A b$ if for every possible dependency set $D^b \in \mathsf{dep}(b)$ such that $D^b \subseteq A$, we have that $a \in D^b$. We say that b **must depend on** a **within** A. Clearly when $A = E$, then $a \lhd_E b$ when a belongs to every depset of b, i.e. b must depend on a.

Lemma 2.25. For any nonempty subset $A \subseteq E$, $\lhd_A$ is a transitive relation. If A is a nonempty complete event set, then $\lhd_A$ is also irreflexive.

Proof. Suppose that $a \lhd_A b$ and $b \lhd_A c$. Now suppose that $X \in \text{dep}(c)$ and $X \subseteq A$. Then $b \in X$, since $b \lhd_A c$. But X is complete, since E is a DSC, so there exists a $D^b \in \text{dep}(b)$ such that $D^b \subseteq X$. But if $D^b \subseteq X$, then $D^b \subseteq A$, so $a \in D^b$ since $a \lhd_A b$, which implies that $a \in X$. Thus $a \lhd_A c$.

Now if $a \lhd_A a$, and A is a nonempty complete event set, then every possible dependency set X of a that is a subset of A must contain a. Now since A is complete, and $a \in A$, there must exist a possible dependency set $X \in \text{dep}(a)$ such that $X \subseteq A$. Thus X contains a, which is not allowed since E is a DSC. Thus $\lhd_A$ is irreflexive. □

Lemma 2.26. Suppose E is a DSC, and $A \subseteq E$ is a nonempty complete event set. Then there exists an $a \in A$ such that $(A \setminus \{a\})$ is complete.

Proof. Suppose not, then $(A \setminus \{a\})$ is not complete for any $a \in A$.

Choose $a_0 \in A$. Then $(A \setminus \{a_0\})$ is not complete. Thus there exists some $a_1 \in (A \setminus \{a_0\})$ such that if $X \in \text{dep}(a_1)$, then $X \not\subseteq (A \setminus \{a_0\})$. However A is complete, and $a_1 \in A$, thus if $X \in \text{dep}(a_1)$, then $X \subseteq A$, so X must contain a_0. Thus $a_0 \lhd_A a_1$.

Now $A \setminus \{a_1\}$ is also not complete, so by the same argument there exists some a_2 such that $a_1 \lhd_A a_2$.

Continuing this, we end up with a sequence

$$a_0 \lhd_A a_1 \lhd_A \cdots \lhd_A a_n,$$

that either enumerates all of the elements of A, or there is a $\lhd_A$-cycles with $a_i = a_j$ for some $0 \le i < j \le n$. Suppose there is a cycle, then by Lemma 2.25, $\lhd_A$ is transitive, so we have $a_k \lhd_A a_k$. However, since A is complete, $\lhd_A$ is irreflexive, so we have a contradiction.

Now suppose the above sequence enumerates all of the elements of A. Then since $A \setminus \{a_n\}$ is not complete by assumption, so there exists some a_k such that $a_n \lhd_A a_k$. Thus there exists a $\lhd_A$-cycle

$$a_k \lhd_A a_{k+1} \lhd_A \cdots \lhd_A a_n \lhd_A a_k,$$

which by the above reasoning cannot happen. Thus there must exist some $a \in A$ such that $(A \setminus \{a\})$ is complete.

□

Proposition 2.27. Let E be a DSC, then a subset $A \subseteq E$ is reachable if and only if it is complete.

Proof. ($\Rightarrow$) Suppose that A is reachable, then there exists a sequence of subsets $X_i \subseteq E$ such that

$$\emptyset \,\mathsf{rch}\, X_1 \,\mathsf{rch}\, X_2 \,\mathsf{rch} \ldots \mathsf{rch}\, X_n \,\mathsf{rch}\, A.$$

Since $X_n \,\mathsf{rch}\, A$, $X_n \subseteq A$, and for every $a \in A$, there exists a possible dependency set $D^a_i \in \text{dep}(a)$ such that $D^a_i \subseteq X_n$. But this implies that $D^a_i \subseteq A$, so A is a complete event set.

($\Leftarrow$) Suppose that A is a complete event set. Then by Lemma 2.26, there exists an $a \in A$ such that $(A \setminus \{a\})$ is complete. Thus if $x \in (A \setminus \{a\})$, then there exists a possible dependency set $D^x \in \text{dep}(x)$ such that $D^x \subseteq (A \setminus \{a\})$. Since A is complete and $a \in A$, there exists a possible dependency set $D^a \in \mathrm{dep}(a)$, such that $D^a \subseteq A$. Since E is a DSC, $a \notin D^a$, thus $D^a \subseteq (A \setminus \{a\})$, thus $(A \setminus \{a\}) \,\mathsf{rch}\, A$.

Now enumerate $A = \{a_0, \ldots, a_n\}$. Then by induction, there exists another ordering on A, written $\{a_{i_0}, \ldots, a_{i_n}\}$ such that

$$\varnothing \,\mathsf{rch}\, (A \setminus \{a_{i_0}, \ldots, a_{i_{n-1}}\}) \,\mathsf{rch}\, \ldots \,\mathsf{rch}\, (A \setminus \{a_{i_0}\}) \,\mathsf{rch}\, A.$$

Thus A is reachable. □

Remark 2.28. For the rest of the paper, we will not distinguish between reachable and complete event sets of a DSC, and will mostly refer to them as complete event sets.

Proposition 2.27 will be very useful in our analysis of DSCs, and also provides us with the key technical property that relates DSCs and antimatroids in Section 3. Let us now analyze the structure of $\mathsf{rdp}(E)$ for a DSC (E, dep) more deeply.

Definition 2.29. Given a join-semilattice L, an element $x \in L$ is **join-irreducible** if $x \neq \bot$ and whenever $a \vee b = x$ then $a = x$ or $b = x$. Let $\mathcal{J}(L)$ denote the subposet of join irreducible elements.

Given a DSC (E, dep), we can characterize the join-irreducible elements of $\mathsf{rdp}(E)$. This will also be important in Section 6.

Definition 2.30. Let (E, dep) be a DSC, $e \in E$, and $X \subseteq E$. Say X is an ***e*-minimal complete event set** if X is complete, $e \in X$ and if for every complete subset $Y \subseteq X$ such that $e \in Y$, it follows that $X = Y$. Let $\mathrm{Min}(e)$ denote the set of e-minimal complete event sets.

Lemma 2.31. Suppose (E, dep) is a DSC and $X \subseteq E$ is a complete event set. If X contains an element e, then there exists an e-minimal complete event set M_e of E such that $M_e \subseteq X$.

Proof. Suppose X is a complete event set and $e \in X$. If there exist no proper subsets of X that contain e and are complete, then X is an e-minimal complete event set. So suppose that X is not an e-minimal complete event set. Then there must be a complete proper subset $X_1 \subset X$ such that X_1 contains e. If X_1 is not an e-minimal complete event set, then there's a complete proper subset $X_2 \subset X_1$ such that X_2 contains e. Since E is finite, there must be a finite descending chain of complete proper subsets

$$\varnothing \subset X_n \subset X_{n-1} \subset \cdots \subset X_1 \subset X$$

which means that X_n must be an e-minimal complete event set. □

Proposition 2.32. If E is a DSC, then for every $e \in E$, an event set $X \subseteq E$ is an e-minimal complete event set if and only if it is of the form $D^e \cup e$, for $D^e \in \mathrm{dep}(e)$.

Proof. First note that if $D^e \in \mathrm{dep}(e)$, then $D^e \cup e$ is a complete event set. Now let us show that $D^e \cup e$ is an e-minimal complete event set. Suppose that X is complete, $X \subseteq D^e \cup e$ and $e \in X$. Then there exists a depset D^e_0 of e such that $D^e_0 \subseteq (X \setminus e) = D^e$, because X is complete, and depsets of e can't contain e. This implies that $D^e_0 \subseteq D^e$, which by (D0) implies $D^e_0 = D^e$, in which case $X = D^e \cup e$.

Now suppose that X is an e-minimal complete event set. Then $e \in X$, and since X is complete, there exists some $D^e \in \mathrm{dep}(e)$ such that $D^e \subseteq X$. Clearly $D^e \cup e \subseteq X$, and thus since X is an e-minimal complete event set, $D^e \cup e = X$. □

Proposition 2.33. Let (E, dep) be a DSC, and consider $\mathsf{rdp}(E)$. If $X \in \mathsf{rdp}(E)$, then X is join-irreducible if and only if X is an e-minimal complete event set for some $e \in E$.

Proof. ($\Leftarrow$) Suppose that X is an e-minimal complete event set and $X = A \cup B$, where A, B are complete. Since $e \in X$, then $e \in A$ or $e \in B$ or both. Without loss of generality suppose that $e \in A$. Then by definition of e-minimal complete event sets, $X = A$. Thus X is join-irreducible.

($\Rightarrow$) Suppose that X is a complete event set of E but not an e-minimal complete event set for any $e \in E$. We wish to show that X is not join-irreducible. If X is not e-minimal for any $e \in E$, then it is not x-minimal for any $x \in X$. Therefore, for every $x \in X$, by Lemma 2.31 there exists an x-minimal complete event set $M_x \subseteq X$. Therefore $X = \cup_{x \in X} M_x$, so X is a join of finitely many complete event sets. Thus X is not join-irreducible. □

Now let us show that $\mathsf{rdp}(E)$ is a lattice. This is a general fact about finite join-semilattices. Indeed, if P is a finite join-semilattice with a bottom element and $X, Y \in P$, then their meet always exists and is given by

$$X \wedge Y = \bigvee \{Z \in P : Z \leq X, Z \leq Y\}. \tag{4}$$

In the case of $\mathsf{rdp}(E)$, this means that the meet of two reachable event sets X and Y will be the union of all those reachable event sets that are a subset of $X \cap Y$.

Lemma 2.34. Let (E, dep) be a DSC, and $A, B \subseteq E$ complete event sets. Then

$$A \wedge B = \{x \in A \cap B : \exists D^e \in \mathrm{dep}(e) \text{ such that } D^e \subseteq A \cap B\}.$$

Proof. Let $C = \{x \in A \cap B : \exists D^e \in \mathrm{dep}(e) \text{ such that } D^e \subseteq A \cap B\}$. Now $C \subseteq A \cap B$. We wish to show that C is complete. So suppose $x \in C$. Then there exists $D^x \in \mathrm{dep}(x)$ such that $D^x \subseteq A \cap B$. If $y \in D^x$, then since D^x is complete, there exists a $D^y \subseteq D^x$ with $D^y \in \mathrm{dep}(y)$. Thus $D^y \subseteq A \cap B$, so $y \in C$. Since y was arbitrary, $D^x \subseteq C$, so C is complete. Thus $C \subseteq A \wedge B$. We wish to prove the opposite inclusion. By (4) if $x \in A \wedge B$ then x belongs to some set $X \subseteq A \cap B$ that is complete, so there exists a $D^x \in\subseteq X$, thus $x \in C$. Thus $A \wedge B = C$. □

Remark 2.35. It is important to notice that while every finite join-semilattice is also a finite lattice, this extension is not functorial.

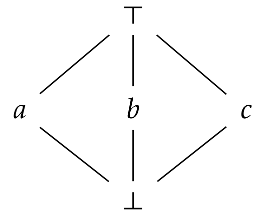
Now let us introduce a well-studied special class of lattices.

Definition 2.36. We say that a lattice L is **distributive** if for every $x, y, z \in L$, the following identity holds:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Distributive lattices enjoy a key characterization theorem due to Birkhoff [Bir40]. To discuss that theorem we need to introduce two important examples of lattices. Consider the following lattice, known as M_3 or the **diamond lattice**, which has five elements, $\bot, a, b, c, \top$, and order structure presented by the Hasse Diagram:

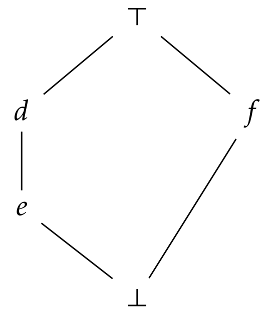
(5)



Note that this lattice is not distributive, indeed $a \wedge (b \vee c) = a \wedge \top = a$, but $(a \wedge b) \vee (a \wedge c) = \bot \vee \bot = \bot$.

Now let N_5 denote the following lattice, also known as the **pentagon lattice**, which also has five elements, $\bot, d, e, f, \top$ with the following order structure:

(6)



Note that this lattice is also not distributive.

Definition 2.37. Given a lattice L, a subset $S \subseteq L$ is called a **sublattice** if for all $x, y \in S$, $x \wedge y \in S$ and $x \vee y \in S$.

Theorem 2.38 ([Bir40])**.** A lattice L is distributive if and only if there exists no sublattice of L that is isomorphic to M_3 or N_5.

Many examples of lattices are indeed distributive. Indeed one can check that if S is a set, then its lattice of subsets $P(S)$ is distributive. However it is typically not the case that if E is a DSC, then $\mathrm{rdp}(E)$ will be a distributive lattice.

Example 2.39. Notice that Example 2.18 is not a distributive lattice, as

$$b \vee (ab \wedge ac) = b \vee \varnothing = b$$

meanwhile

$$(b \vee ab) \wedge (b \vee ac) = ab \wedge abc = ab.$$

In other words $\{\varnothing, b, ab, ac, abc\}$ is a sublattice of $a.b \vee c$ that is isomorphic to N_5. In fact it contains two such sublattices, the other being $\{\varnothing, c, ac, ab, abc\}$.

We will see in Section 3 that while the lattices in the image of rdp are not all distributive, they satisfy a weaker condition of diamond-free semimodularity, which is to say, they correspond to antimatroids [Czé14]. Thus they will not contain a copy of the diamond lattice M_3, but they could possibly contain the pentagon lattice N_5. Further, in Section 4, we will show that there is a subclass of DSCs whose image under rdp does land in distributive lattices.

2.3. **Morphisms.** In [Baz21], the first author defined morphisms of DSCs to be certain kinds of relations. In this section we will define a more restricted class of morphisms which are precisely those morphisms needed to make the rdp construction functorial.

Let (E, dep) and (E', dep') be DSCs, and let $f : E \to E'$ be a function between their underlying sets. Then the preimage map $f^* : P(E') \to P(E)$ can be restricted to a map from only reachable event sets, $f^* : \mathsf{rdp}(E') \to P(E)$. We wish to consider the class of functions f such that f^* factors through $\mathsf{rdp}(E)$, which is equivalent to asking that f^* preserves reachable event sets.

Definition 2.40. Let (E, dep) and (E', dep') be DSCs and suppose $f : E \to E'$ is a function between their underlying sets. We say that f is a morphism of DSCs if for all $e \in E$, and all $D^{f(e)} \in \mathrm{dep}(f(e))$, there exists a $D^e \in \mathrm{dep}(e)$ such that $D^e \subseteq f^*\left(D^{f(e)} \cup f(e)\right)$. We will often refer to morphisms of DSCs as maps of DSCs.

Remark 2.41. By Proposition 2.32 the above condition is equivalent to the following: f is a morphism of DSCs if for every $e \in E$ and $f(e)$-minimal complete event set $M_{f(e)}$, then the preimage $f^*(M_{f(e)})$ contains an e-minimal complete event set.

It is not hard to check that the composition of two maps of DSCs is a map of DSCs, and the identity function is a map of DSCs. Thus DSCs with these morphisms form a category which we will denote by DSC.

Lemma 2.42. Let $f : E \to E'$ be a function of the underlying sets of DSCs. Then f is a map of DSCs if and only if f^* preserves reachable sets.

Proof. ($\Rightarrow$) By Proposition 2.27, it is equivalent to show that f^* preserves completeness. Suppose that $A \subseteq E'$ is complete. We wish to show that $f^*(A)$ is complete. Suppose $e \in f^*(A)$. Then $f(e) \in A$, but A is complete so there exists a $D^{f(e)} \in \mathrm{dep}(f(e))$ such that $D^{f(e)} \subseteq A$. Since f is a map of DSCs, there exists a $D^e \in \mathrm{dep}(e)$ such that $D^e \subseteq f^*(D^{f(e)} \cup f(e))$. This implies that $D^e \subseteq f^*(A)$, and thus $f^*(A)$ is complete.

($\Leftarrow$) Suppose that f^* preserves reachable/complete event sets, $e \in E$ and $D^{f(e)} \in \mathrm{dep}(f(e))$. Then $D^{f(e)} \cup f(e)$ is complete, so $f^*\left(D^{f(e)} \cup f(e)\right)$ is complete and contains e, thus there exists some $D^e \in \mathrm{dep}(e)$ such that $D^e \subseteq f^*\left(D^{f(e)} \cup f(e)\right)$. Thus f is a map of DSCs. □

Corollary 2.43. If $f : E \to E'$ is a set function between DSCs, then f is a morphism of DSCs if and only if f^* descends to an order preserving map of posets $f^* : \mathsf{rdp}(E') \to \mathsf{rdp}(E)$. Further, as joins are given by unions, and preimage preserves unions, this is a map of join-semilattices.

Thus we can summarize the above by saying that if $\mathsf{FinJSLatt}$ denotes the category whose objects are finite join-semilattices and whose morphisms are order-preserving and join-preserving functions, then the rdp construction extends to a functor

$$\mathsf{rdp} : \mathsf{DSC} \to \mathsf{FinJSLatt}^{op},$$

where if $f:(E,\mathrm{dep})\to(E',\mathrm{dep}')$ is a map of DSCs, then we set $\mathsf{rdp}(f)=f^*$.

Lemma 2.42 is the first indication that maps of DSCs as given in Definition 2.40 are natural to use, as they are the largest class of morphisms for which rdp is functorial. The connection between antimatroids and DSCs that will be discussed in Section 3 will provide another such indication.

Definition 2.44. Suppose $f:E\to E'$ is a set function of the underlying sets of DSCs. We say that f is a comorphism if for all $e\in E$, and all $D^e\in\mathrm{dep}(e)$, there exists a $D^{f(e)}\in\mathrm{dep}(f(e))$ such that $D^{f(e)}\subseteq f_*(D^e\cup e)$.

Lemma 2.45. Let $f:E\to E'$ be a function of the underlying sets of DSCs. Then f is a comorphism if and only if f_* preserves reachable sets.

Proof. ($\Rightarrow$) By Proposition 2.27, it is equivalent to show that f_* preserves completeness. Suppose that $A\subseteq E$ is complete, we want to show that $f_*(A)$ is complete. Suppose $y\in f_*(A)$. Then there exists an $e\in A$ such that $y=f(e)$. Since A is complete there exists a $D^e\in\mathrm{dep}(e)$, such that $D^e\subseteq A$. Since f is a comorphism, there exists a $D^{f(e)}\in\mathrm{dep}(f(e))$ such that $D^{f(e)}\subseteq f_*(D^e\cup e)$. This implies that $D^{f(e)}\subseteq f_*(A)$. Thus $f_*(A)$ is complete.

($\Leftarrow$) Suppose that f_* preserves reachable/complete event sets, $e\in E$ and $D^e\in\mathrm{dep}(e)$. Then $D^e\cup e$ is complete, so $f_*(D^e\cup e)$ is complete and contains $f(e)$, thus there exists a $D^{f(e)}\in\mathrm{dep}(f(e))$ such that $D^{f(e)}\subseteq f_*(D^e\cup e)$. Thus f is a comorphism ☐

Corollary 2.46. If f is a comorphism, then f_* descends to an order preserving map of posets $f_*:\mathsf{rdp}(E)\to\mathsf{rdp}(E')$. Further, as joins are given by unions, this is a map of join-semilattices.

Now we wish to define a category DSC' whose objects are DSCs, and with morphisms such that rdp is a functor of type $\mathsf{rdp}:\mathsf{DSC}'\to\mathsf{FinLatt}^{op}$, where FinLatt is the category whose objects are finite lattices and whose morphisms are order, join and meet-preserving functions. In other words, we wish to find a class of set functions f such that $\mathsf{rdp}(f)=f^*$ preserves reachable event sets and is a lattice morphism, i.e. preserves joins and meets. We have already established join preservation, and now turn our attention to meets[2].

Lemma 2.47. If $f:E\to E'$ is a morphism and comorphism of DSCs, then $f^*:\mathsf{rdp}(E')\to\mathsf{rdp}(E)$ preserves meets.

Proof. We want to show that $f^*(A\wedge B)=f^*(A)\wedge f^*(B)$. Since $A\wedge B\subseteq A\cap B$, we have $f^*(A\wedge B)\subseteq f^*(A\cap B)=f^*(A)\cap f^*(B)$. But $f^*(A\wedge B)$ is reachable since f is a map, and $f^*(A)\wedge f^*(B)$ is the largest reachable subset of $f^*(A)\cap f^*(B)$, so $f^*(A\wedge B)\subseteq f^*(A)\wedge f^*(B)$.

Let us show that $f^*(A\wedge B)$ is the largest reachable subset of $f^*(A\cap B)$, as this will imply the result. Suppose that $X\subseteq f^*(A\cap B)$ is a reachable subset. Then $f_*(X)\subseteq f_*f^*(A\cap B)\subseteq A\cap B$. Since f is a comorphism, $f_*(X)$ is reachable, so $f_*(X)\subseteq A\wedge B$. Thus $X\subseteq f^*f_*(X)\subseteq f^*(A\wedge B)$. Thus $f^*(A\wedge B)$ is the largest reachable subset of $f^*(A\cap B)$. ☐

Definition 2.48. We say that $f:(E,\mathrm{dep})\to(E',\mathrm{dep}')$ is a bimorphism of DSCs if it is both a morphism and comorphism of DSCs.

[2]We only consider f^* rather than f_* here because the latter does not in general preserve intersections of sets.

Corollary 2.49. Let DSC' denote the category whose objects are DSCs and whose morphisms are bimorphisms. Then $\mathsf{rdp}: \mathsf{DSC}' \to \mathsf{FinLatt}^{op}$ is a functor.

Later in Section 6, we will have use for a still more restricted class of morphisms.

Definition 2.50. Let E, E' be DSCs. A **distributive-preserving map** of DSCs is a set function $f : E \to E'$ such that:

(1) f is a bimorphism of DSCs, and
(2) $f^* : \mathsf{rdp}(E') \to \mathsf{rdp}(E)$ is surjective.

Remark 2.51. The second condition implies that f^* will preserve distributive families, which we will explain the importance of in Section 6.

Lemma 2.52. If $f : E \to E'$ is a bimorphism of DSCs and is injective, then f is a distributive-preserving morphism of DSCs.

Proof. If f is injective, then $f^* f_*(X) = X$ for all $X \subseteq E$. Thus considering $f^* : \mathsf{rdp}(E') \to \mathsf{rdp}(E)$, we want to show that if $X \in \mathsf{rdp}(E)$, then $X = f^* Y$ for some $Y \in \mathsf{rdp}(E)$. By Lemma 2.45, if $X \subseteq E$ is reachable, then $f_*(X)$ is reachable, and $f^* f_*(X) = X$, thus f^* is surjective. □

3. Antimatroids and DSCs

In this section we will establish an isomorphism between the category of antimatroids and the category of DSCs. Further, we will use this to characterize the essential image of the functor $\mathsf{rdp}: \mathsf{DSC} \to \mathsf{FinJSLatt}^{op}$ as finite diamond-free semimodular lattices.

3.1. **Antimatroids.**

Antimatroids are a mathematical construction notable for their frequent rediscovery, as chronicled in [Mon85]. Our work here is another instance of that rediscovery. First studied by Dilworth in 1940 [Dil40], these arise in characterizing greedy algorithms [BF90], combinatorics of convex spaces [Czé14], and artificial intelligence planning [Par03], among other applications.

Definition 3.1. An **antimatroid** is a pair $(E, \mathcal{F})$, where E is a finite set and $\mathcal{F} \subseteq P(E)$ is a nonempty family of subsets, called the **feasible subsets** of E, such that:

(A1) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$, and
(A2) if $S \neq \emptyset$ and $S \in \mathcal{F}$, then there exists an element $x \in S$ such that $(S \setminus x) \in \mathcal{F}$,
(A3) $E = \bigcup_{A \in \mathcal{F}} A$.

Remark 3.2. Condition (A3) in Definition 3.1 is sometimes omitted in the literature, and antimatroids that satisfy (A3) are called **shelling structures** such as in [KL85]. In [Czé14] however they are just referred to as antimatroids. It makes little difference in the theory as one could always just consider the resulting antimatroid $(\bigcup_{A \in \mathcal{F}} A, \mathcal{F})$.

Example 3.3. If E is a finite set, then the pair $(E, P(E))$ is an antimatroid, which we call the **maximal antimatroid** of E.

Definition 3.4. Let $(E, \mathcal{F}), (E', \mathcal{F}')$ be antimatroids and $f : E \to E'$ a set function such that if $A \in \mathcal{F}'$, then $f^*(A) \in \mathcal{F}$. We call f a morphism of antimatroids and denote it by $f : (E, \mathcal{F}) \to (E', \mathcal{F}')$.

Remark 3.5. There are several notions of morphisms of antimatroids in the literature, with most sources not defining morphisms at all. The above definition is a restriction of the definition of morphism of greedoids to antimatroids as given in [XS15, Definition 4]. We will show why this class of morphisms is appropriate for our considerations.

It is easy to see that morphisms of antimatroids are closed under composition and the identity function is a morphism of antimatroids, thus we can define the category AntiMat whose objects are antimatroids and whose morphisms are morphisms of antimatroids.

An important class of examples of antimatroids comes from partially ordered sets.

Definition 3.6. Given a poset P, a subset $A \subseteq P$ is called a **downset** if for every $x, y \in P$ such that $y \leq x$ and $x \in A$, we have that $y \in A$. Consider the set $\mathcal{O}(P)$ of downsets of P. This is clearly a poset by inclusion, and it is not hard to show that $\mathcal{O}(P)$ is a distributive lattice. This extends to a functor $\mathcal{O} : \mathsf{FinPos} \to \mathsf{FinDLatt}^{op}$, by defining $\mathcal{O}(f) : \mathcal{O}(P') \to \mathcal{O}(P)$ by $X \mapsto f^*(X)$, for a map $f : P \to P'$ of posets.

Example 3.7. If P is a finite poset, it is not hard to show that the pair $(P, \mathcal{O}(P))$ is an antimatroid. We call antimatroids of this form **poset antimatroids**. These are a well-studied class of antimatroids, and are the unique such class whose feasible sets are closed under intersections, see [KL13]. We will consider the connection between certain kinds of DSCs and this class of antimatroids in Section 4.

We wish to show that the construction of poset antimatroids forms a functor $D : \mathsf{FinPos} \to \mathsf{AntiMat}$. Indeed, if $f : P \to P'$ is an order preserving map of posets, then let us show that $D(f) : (P, \mathcal{O}(P)) \to (P', \mathcal{O}(P'))$ is a morphism of antimatroids. If V is a downset of P', we wish to show that $f^*(V)$ is a downset of P. Suppose that $x \in f^*(V)$ and $y \in P$ such that $y \leq x$. Then since f is order-preserving, $f(y) \leq f(x)$, and since V is a downset, $f(y) \in V$, thus $y \in f^*(V)$. Thus $f^*(V)$ is a downset of P.

Lemma 3.8. The functor $D : \mathsf{FinPos} \to \mathsf{AntiMat}$ is fully faithful.

Proof. It is easy to see that if $P, P' \in \mathsf{FinPos}$, then the canonical function $D : \mathsf{FinPos}(P, P') \to \mathsf{AntiMat}((P, \mathcal{O}(P)), (P', \mathcal{O}(P')))$ is injective, since $D(f)$ is just f on the underlying sets. We wish to show that it is also surjective. So suppose that $f : P \to P'$ is a function such that it pulls downsets of P' back to downsets of P. We wish to show that it is also order preserving. Suppose that $y \leq x$ in P. Then $y \in \downarrow x$, and $f^*(\downarrow f(x))$ is a downset containing x, therefore $\downarrow x \subseteq f^*(\downarrow f(x))$. Thus $f(y) \in \downarrow f(x)$, so $f(y) \leq f(x)$. Thus D is fully faithful. □

3.2. **Correspondence of Antimatroids and DSCs.**

In this section we prove one of the main theorems of this paper, Theorem 3.15.

Proposition 3.9. If E is a DSC, then $(E, \mathsf{rdp}(E))$ is an antimatroid.

Proof. By Lemma 2.22, reachable subsets are closed under unions, proving condition (A1) of Definition 3.1. Proposition 2.27 and Lemma 2.26 prove (A2). Since E is complete, this proves (A3). □

Given a DSC (E, dep), let $\Phi(E, \mathsf{dep}) = (E, \mathsf{rdp}(E))$ denote its corresponding antimatroid. We wish to show that this construction defines a functor $\Phi : \mathsf{DSC} \to \mathsf{AntiMat}$. Suppose that $f : E \to E'$ is a map of DSCs in the sense of Definition 2.40. Thus

f^* preserves reachable subsets by Lemma 2.42, so f becomes a map of antimatroids $f : (E, \mathsf{rdp}(E)) \to (E', \mathsf{rdp}(E'))$, because if $X \in \mathsf{rdp}(E')$, then $f^*(X) \in \mathsf{rdp}(E)$. It is now easy to check that this makes $\Phi : \mathsf{DSC} \to \mathsf{AntiMat}$ into a functor.

We wish to now construct a functor from the category of antimatroids to the category of DSCs.

Definition 3.10. Let $(E, \mathcal{F})$ be an antimatroid, and $e \in E$. We say that a feasible subset $X \subseteq E$ is an e-**minimal feasible set** if $e \in X$ and for all feasible sets Y such that $e \in Y$ and $Y \subseteq X$, it follows that $X = Y$. Let $\mathrm{Min}_{\mathcal{F}}(e)$ denote the set of e-minimal feasible sets.

Lemma 3.11. Suppose $(E, \mathcal{F})$ is an antimatroid and $X \in \mathcal{F}$ is a feasible set. If X contains an element e, then there exists an e-minimal feasible set $M_e \subseteq X$.

Proof. This follows by essentially the same proof as Lemma 2.31. □

Now suppose $(E, \mathcal{F})$ is an antimatroid. Consider the preDSC $(E, \mathrm{dep}_{\mathcal{F}})$, where $\mathrm{dep}_{\mathcal{F}} : E \to P(P(E))$ is defined by $\mathrm{dep}_{\mathcal{F}}(e) = \{(M_e \setminus e) : M_e \in \mathrm{Min}_{\mathcal{F}}(e)\}$. We wish to show that $(E, \mathrm{dep}_{\mathcal{F}})$ is a DSC. We check the axioms of Definition 2.9.

- (D0) If $X, Y \in \mathrm{dep}_{\mathcal{F}}(e)$, and $X \subseteq Y$, then since $X \cup e$ and $Y \cup e$ are e-minimal feasible sets, we have that $X = Y$.
- (D1) By (A3), we know that every $e \in E$ must be contained in some feasible set, and therefore must be contained in some e-minimal feasible set by Lemma 3.11.
- (D2) If $X \in \mathrm{dep}_{\mathcal{F}}(e)$, then X is of the form $M_e \setminus e$, therefore $e \notin X$.

Now to prove (D3), let us show that each depset $M_e \setminus e$ is complete. First we need the following result.

Lemma 3.12. Suppose that $(E, \mathcal{F})$ is an antimatroid, $e \in E$, and M_e is an e-minimal feasible set. If $x \in (M_e \setminus e)$, then there exists an x-minimal feasible set $M_x \subseteq (M_e \setminus e)$.

Proof. Since $x \in M_e$, by Lemma 3.11 there exists an x-minimal feasible set $M_x \subseteq M_e$. Suppose that M_e contains no feasible strict subsets that contain x, so that $M_x = M_e$. By (A2) there exists a $y \in M_e$ such that $M_e \setminus y$ is feasible. If $y \neq x$, then $(M_e \setminus y) \subset M_x$, so $(M_e \setminus y)$ is a feasible set containing x, which is a contradiction. If $y = x$, then $(M_e \setminus x)$ is feasible, but M_e is an e-minimal feasible set, so $(M_e \setminus x) = M_e$, which is also a contradiction. Thus $M_x \subset M_e$. If M_x contains e, then since M_e is an e-minimal feasible set, $M_x = M_e$, which is a contradiction. Thus $M_x \subseteq (M_e \setminus e)$. □

So if $x \in (M_e \setminus e)$, then by Lemma 3.12, there exists an x-minimal feasible set $M_x \subseteq (M_e \setminus e)$. Thus $(M_x \setminus x) \subseteq (M_e \setminus e)$. So $(M_e \setminus e)$ contains a depset of x. Thus every depset is complete, so it satisfies (D3). Thus we have proven the following.

Proposition 3.13. Given an antimatroid $(E, \mathcal{F})$, the corresponding preDSC $(E, \mathrm{dep}_{\mathcal{F}})$ is a DSC.

Given an antimatroid $(E, \mathcal{F})$, let $\Psi(E, \mathcal{F}) = (E, \mathrm{dep}_{\mathcal{F}})$. We wish to show that Ψ extends to a functor $\Psi : \mathsf{AntiMat} \to \mathsf{DSC}$. Suppose $f : (E, \mathcal{F}) \to (E', \mathcal{F}')$ is a morphism of antimatroids. We want to show that f now defines a morphism of DSCs $f : (E, \mathrm{dep}_{\mathcal{F}}) \to (E', \mathrm{dep}_{\mathcal{F}'})$. In other words by Lemma 2.42, we need to show that for every $e \in E$, and $f(e)$-minimal complete event set $M_{f(e)}$ in $(E', \mathrm{dep}_{\mathcal{F}'})$, there exists an e-minimal complete event set $M_e \subseteq f^*(M_{f(e)})$. But $f(e)$-minimal complete event sets in $(E', \mathrm{dep}_{\mathcal{F}'})$ are of the form $D^{f(e)} \cup f(e)$ by Proposition 2.32, where $D^{f(e)} \in \mathrm{dep}_{\mathcal{F}}(f(e))$. But every $D^{f(e)}$ is of the form $N_{f(e)} \setminus f(e)$ for an $f(e)$-minimal feasible set $N_f(e) \in \mathcal{F}'$. In other words, every $f(e)$-minimal complete event set $M_{f(e)}$ of $(E', \mathrm{dep}'_{\mathcal{F}})$ is also an $f(e)$-minimal feasible set in $(E', \mathcal{F}')$. Since f is a morphism of antimatroids, if $M_{f(e)}$ is an $f(e)$-minimal feasible set in $(E', \mathcal{F}')$, then $f^*(M_{f(e)})$ will be a feasible set in $\mathcal{F}$ which contains e. Thus there exists an e-minimal complete event set $M_e \subseteq f^*(M_{f(e)})$. This implies that f is a map of DSCs, and therefore Ψ defines a functor.

Now consider the two functors $\Psi : \mathsf{AntiMat} \to \mathsf{DSC}$ and $\Phi : \mathsf{DSC} \to \mathsf{AntiMat}$. If (E, dep) is a DSC, then Proposition 2.32 implies that

$$(\Psi \circ \Phi)(E, \mathrm{dep}) = (E, \mathrm{dep}).$$

Now suppose $(E, \mathcal{F})$ is an antimatroid. We want to show that the other composition is the identity. Consider $(\Phi \circ \Psi)(E, \mathcal{F}) = \Phi(E, \mathrm{dep}_{\mathcal{F}}) = (E, \mathsf{rdp}(E, \mathrm{dep}_{\mathcal{F}}))$. It is sufficient to prove the following.

Lemma 3.14. Let $(E, \mathcal{F})$ be an antimatroid, and consider $\Psi(E, \mathcal{F}) = (E, \mathrm{dep}_{\mathcal{F}})$. An event set X of $(E, \mathrm{dep}_{\mathcal{F}})$ is complete if and only if X is a feasible set of $(E, \mathcal{F})$.

Proof. ($\Leftarrow$) If $X \in \mathcal{F}$ is feasible, then by Lemma 3.11, X is a complete event set in $(E, \mathrm{dep}_{\mathcal{F}})$.

($\Rightarrow$) Suppose that $X \subseteq E$ is complete. Then for each $x \in X$, there exists a depset $(M_x \setminus x) \subseteq X$. Of course this means $M_x \subseteq X$. Thus $\bigcup_{x \in X} M_x \subseteq X$ and $X \subseteq \bigcup_{x \in X} M_x$. Since unions of feasible sets are feasible, X is feasible. ☐

Thus $\mathsf{rdp}(E, \mathrm{dep}_{\mathcal{F}})$ consists of precisely the feasible sets of $(E, \mathcal{F})$, namely $\mathsf{rdp}(E, \mathrm{dep}_{\mathcal{F}}) = \mathcal{F}$. Thus we have proven that Φ and Ψ induce a bijection between antimatroids and DSCs. Since both of these functors act as the identity on morphisms, the following result follows.

Theorem 3.15. The functors $\Phi : \mathsf{DSC} \to \mathsf{AntiMat}$ and $\Psi : \mathsf{AntiMat} \to \mathsf{DSC}$ defined above form an isomorphism of categories

$$\mathsf{AntiMat} \cong \mathsf{DSC}.$$

3.3. **Semimodular Lattices.** Here we review the definitions of modular and semimodular lattices and relate them to DSCs.

Given a lattice L and elements $a, b \in L$, consider the sublattice $[a, b] = \{x \in L : a \leq x \leq b\}$. Then there is an adjoint pair

$$a \vee (-) : [a \wedge b, b] \rightleftarrows [a, a \vee b] : (-) \wedge b,$$

which is equivalent to saying that whenever $a \wedge b \leq x \leq b$, then $x \leq (a \vee x) \wedge b$ and whenever $a \leq y \leq a \vee b$, then $a \vee (y \wedge b) \leq y$, which is easy to verify.

Definition 3.16. We say that a lattice L is **modular** if the above is an adjoint equivalence for all $a, b \in L$, i.e. if whenever $a \wedge b \leq x \leq b$, $x = (a \vee x) \wedge b$ and whenever $a \leq y \leq a \vee b$, it follows that $a \vee (y \wedge b) = y$.

Now recall the pentagon lattice N_5 from (6).

Lemma 3.17 ([Ded00]). A lattice L is modular if and only if it does not contain N_5 as a sublattice.

Definition 3.18. A lattice L is said to be **upper semimodular** if whenever $a \wedge b \lessdot a$, then $b \lessdot a \vee b$, where $\lessdot$ is the cover relation, namely $x \lessdot y$ if $x < y$ and whenever $x \leq z \leq y$, then $z = x$ or $z = y$. We say it is **lower semimodular** if whenever $b \lessdot a \vee b$ then $a \wedge b \lessdot a$.

Remark 3.19. If we refer to a lattice L as **semimodular**, that means L is upper semimodular.

Definition 3.20. A lattice L is a **diamond-free semimodular lattice** (also known as a **join-distributive lattice**) if it is semimodular and does not contain M_3 as a sublattice.

Let $(E, \mathcal{F})$ denote an antimatroid as in Definition 3.1. Since $\mathcal{F} \subseteq P(E)$, it inherits a poset structure. Since $\mathcal{F}$ is closed under unions, $\mathcal{F}$ is actually a join-semilattice. Since it is finite and has a bottom element, one can also define meets as in (4). In fact we have the following characterization:

Proposition 3.21 ([Czé14, Corollary 7.5.i], originally [Ede80, p. 3.3])**.** If $(E, \mathcal{F})$ is an antimatroid, then $(\mathcal{F}, \subseteq)$ is a diamond-free semimodular lattice.

In fact [Czé14, Corollary 7.5] proves that every diamond-free semimodular lattice is of the form $(\mathcal{F}, \subseteq)$ for some antimatroid $(E, \mathcal{F})$.

Corollary 3.22. If (E, dep) is a DSC, then $\mathsf{rdp}(E)$ is a finite diamond-free semimodular lattice. Conversely, every finite diamond-free semimodular lattice is of the form $\mathsf{rdp}(E)$ for some DSC.

In the next section we will pinpoint those DSCs E where $\mathsf{rdp}(E)$ is not only diamond-free semimodular but is also distributive.

4. Dependency Structures with No Choice

In this section we turn to a special class of DSCs, namely those without any choice of dependency sets per event, and show that the essential image of rdp when restricted to DSNCs is equivalent to the opposite of the category of finite distributive lattices.

Definition 4.1. Suppose (E, dep) is a DSC. We say that (E, dep) is a **dependency structure with no choice** or DSNC if each event has a unique possible dependency set, i.e. $\text{dep}(e) = \{D^e\}$ for every $e \in E$. We will abuse notation and write $\text{dep}(e)$ to mean the unique dependency set D^e. Let DSNC denote the full subcategory of DSC whose objects are DSNCs.

For a DSNC (E, dep), and $a, b \in E$ we say that $b \leq a$ if $b \in \text{dep}(a)$. This defines a transitive relation, because if $b \leq a$ and $c \leq b$, then $b \in \text{dep}(a)$ and $c \in \text{dep}(b)$, but $\text{dep}(a)$ is complete, so $\text{dep}(b) \subseteq \text{dep}(a)$, thus $c \in \text{dep}(a)$, so $c \leq a$. Abusing notation, let $\leq$ also denote the closure of this relation under reflexivity, so that $b \leq a$ if $b \in \text{dep}(a)$ or $a = b$. Now if $b \leq a$ and $a \leq b$, then it must be the case that $a = b$, because it cannot be the case that $a \in \text{dep}(a)$. Thus we have defined a reflexive, anti-symmetric, transitive relation on the set E. Denote E with this relation by $(R(E), \leq)$ or just $R(E)$.

We wish to show that this construction extends to a functor $R : \mathsf{DSNC} \to \mathsf{FinPos}$, the codomain of which is the category of finite posets with order preserving maps. Suppose that $f : E \to E'$ is a map of DSNCs. We wish to show that f preserves the ordering.

So suppose that $b \leq a$. This is equivalent to saying that $b \in \mathrm{dep}(a)$. Since f is a map of DSCs, this implies that $\mathrm{dep}(a) \subseteq f^*(\mathrm{dep}(f(a)) \cup f(a))$. Thus $b \in f^*(\mathrm{dep}(f(a))) \cup f^*(f(a))$. If $b \in f^*(f(a))$, then $f(b) = f(a)$, so $f(b) \leq f(a)$. If $b \in f^*(\mathrm{dep}(f(a)))$, then $f(b) \in \mathrm{dep}(f(a))$, which shows that $f(b) \leq f(a)$. Thus f preserves the order of $R(E)$. Further, it is not hard to see that composition and identities are preserved, and thus R defines a functor.

Example 4.2. A simple example of a DSNC is $a.b$, the DSC with $E = \{a, b\}$ and $\mathrm{dep}(a) = \{\{b\}\}$. Its corresponding poset is $R(E) = \{b \leq a\}$.

Lemma 4.3. Let E be a DSNC, then a subset $U \subseteq E$ is complete if and only if U is a downset in $R(E)$.

Proof. ($\Leftarrow$) Let $x \in U$, and $y \in \mathrm{dep}(x)$. Then $y \leq x$ in $R(E)$ by definition. But U is a downset, so $y \in U$. Thus $\mathrm{dep}(x) \subseteq U$, so U is a complete event set.

($\Rightarrow$) Suppose that $U \subseteq E$ is complete, $x \in U$ and $y \leq x$ in R. Then $y \in \mathrm{dep}(x)$, so $y \in U$ since U is complete. Thus U is a downset in $R(E)$. □

This proves the following result.

Lemma 4.4. Let (E, dep) denote a DSNC. Then

$$(\mathcal{O} \circ R)(E) = \mathsf{rdp}(E).$$

We wish to show that $\mathsf{rdp}|_{\mathsf{DSNC}} = \mathcal{O} \circ R$ as functors. Namely suppose $f : (E, \mathrm{dep}) \to (E', \mathrm{dep}')$ is a map of DSNCs. We wish to show that $(\mathcal{O} \circ R)(f) = \mathsf{rdp}(f)$. However, $\mathcal{O} \circ R : \mathsf{DSNC} \to \mathsf{FinDLatt}^{op}$, so $(\mathcal{O} \circ R)(f)$ must be a map of distributive lattices, i.e. it must preserve joins and meets. By Lemma 2.42, we know that $\mathsf{rdp}(f) = f^*$ will preserve joins, but not necessarily meets.

Lemma 4.5. Let (E, dep) be a DSNC, and let $A, B \subseteq E$ be complete event sets. Consider their meet $A \wedge B$ in $\mathsf{rdp}(E)$, then

$$A \wedge B = A \cap B.$$

Proof. From Lemma 2.34, we know that $A \wedge B = \{x \in A \cap B : \mathrm{dep}(x) \subseteq A \cap B\}$. However, since A and B are complete, if $x \in A \cap B$, then we know that $\mathrm{dep}(x) \subseteq A$ and $\mathrm{dep}(x) \subseteq B$, thus $\mathrm{dep}(x) \subseteq A \cap B$. Thus $x \in A \wedge B$. Thus $A \cap B \subseteq A \wedge B$, and obviously $A \wedge B \subseteq A \cap B$. Thus $A \cap B = A \wedge B$. □

Corollary 4.6. Let $f : (E, \mathrm{dep}) \to (E', \mathrm{dep}')$ be a map of DSNCs. Then $\mathsf{rdp}(f) : \mathsf{rdp}(E') \to \mathsf{rdp}(E)$ is a map of finite distributive lattices, namely it preserves meets.

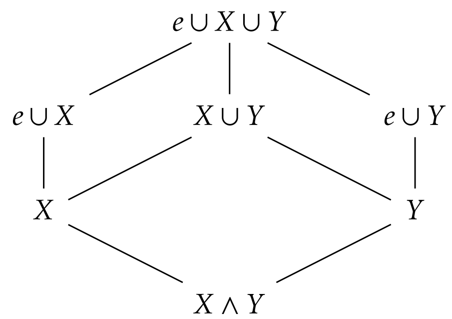
Proof. We wish to show that $f^*(A \wedge B) = f^*(A) \wedge f^*(B)$ for every pair of complete events sets in E'. By Lemma 4.5, $A \wedge B = A \cap B$, and $f^*(A) \wedge f^*(B) = f^*(A) \cap f^*(B)$. Since preimage preserves intersection, the result follows. □

Proposition 4.7. The reachable dependency poset functor when restricted to DSNCs is precisely the composition $\mathcal{O} \circ R$, which is to say the following diagram strictly commutes:

$$\begin{array}{ccc} \mathsf{DSNC} & \hookrightarrow & \mathsf{DSC} \\ {\scriptstyle R}\downarrow & & \\ \mathsf{FinPos} & & \downarrow{\scriptstyle \mathsf{rdp}} \\ {\scriptstyle \mathcal{O}}\downarrow & & \\ \mathsf{FinDLatt}^{op} & \hookrightarrow & \mathsf{FinJSLatt}^{op} \end{array}$$

Now suppose that E is a DSC that is not a DSNC. In other words there exists some $e \in E$ such that $\mathrm{dep}(e)$ is not a singleton. We claim that in this case $\mathsf{rdp}(E)$ cannot be a distributive lattice.

Indeed suppose that $X, Y \in \mathrm{dep}(e)$ are distinct elements. We wish to show that $\mathsf{rdp}(E)$ cannot be distributive. This is easy to see, as the following will have to be a sublattice of $\mathsf{rdp}(E)$.



This sublattice of $\mathsf{rdp}(E)$ contains two copies of the pentagon lattice N_5 as discussed in Example 2.39. Thus if E is a DSC, then $\mathsf{rdp}(E)$ is a distributive lattice if and only if E is a DSNC.

Now consider (one half of) the isomorphism $\Phi : \mathsf{DSC} \to \mathsf{AntiMat}$ from Theorem 3.15. If we restrict Φ to the full subcategory DSNC, then by Proposition 4.7, we see that if $(E, \mathrm{dep}) \in \mathsf{DSNC}$, then $\Phi(E, \mathrm{dep})$ will be a poset antimatroid, as defined in Example 3.7. Similarly restricting Ψ to the full subcategory of poset antimatroids, we see that if $(P, \mathcal{O}(P))$ is a poset antimatroid, and $p \in P$, then $\downarrow p$ will be the unique p-minimal feasible set. Thus $\Psi(P, \mathcal{O}(P)) = (P, \mathrm{dep}_{\mathcal{O}(P)})$ is a DSNC, with $\mathrm{dep}_{\mathcal{O}(P)}(p) = (\downarrow p \setminus p)$. Thus we have proven the following result.

Corollary 4.8. The functors Φ, Ψ of Theorem 3.15 restrict to an isomorphism between the full subcategories of DSNCs and poset animatroids.

Further, recall the functor $D : \mathsf{FinPos} \to \mathsf{AntiMat}$ from Lemma 3.8. By Proposition 4.7 we have $\Phi|_{\mathsf{DSNC}} = D \circ R$. Note that by Lemma 3.8, if we restrict the codomain of D to the full subcategory of poset antimatroids, then D is an equivalence of categories. Since $\Phi|_{\mathsf{DSNC}}$ is an equivalence of categories, this implies the following.

Corollary 4.9. The functor $R : \mathsf{DSNC} \to \mathsf{FinPos}$ is an equivalence of categories,

$$\mathsf{DSNC} \simeq \mathsf{FinPos}.$$

By Birkhoff's Theorem [Bir40], the functor $\mathcal{O} : \mathsf{FinPos} \to \mathsf{FinDLatt}^{op}$ is an equivalence of categories. This implies the following result.

Corollary 4.10. The functor $\mathsf{rdp}|_{\mathsf{DSNC}} = (\mathcal{O} \circ R) : \mathsf{DSNC} \to \mathsf{FinDLatt}^{op}$ is an equivalence of categories.

We can now obtain a similar result for an even smaller class of DSCs.

Definition 4.11. Let (E, dep) be a DSC such that for every $e \in E$, $\text{dep}(e) = \{\varnothing\}$. We say that (E, dep) is a **discrete DSC**.

Remark 4.12. Notice that every discrete DSC is in particular a DSNC.

Let us now show that the functors Φ and Ψ restrict to a bijection between discrete DSCs, and the maximal antimatroids of Example 3.3.

If (E, dep_E) is a discrete DSC, then every subset $X \subseteq E$ is complete. Thus $\Phi(E, \text{dep}_E)$ is clearly a maximal antimatroid. Conversely if $(E, P(E))$ is a maximal antimatroid, and $e \in E$, then $\{e\}$ is the unique e-minimal feasible set. Thus $\Psi(E, P(E))$ is a discrete DSC. It is easy to see that this proves the following result.

Corollary 4.13. The functors Φ, Ψ restrict to define an isomorphism of categories between the full subcategories of discrete DSCs and maximal antimatroids.

5. The Category of DSCs and Antimatroids

In this section we study exactness properties of the category of DSCs and by Theorem 3.15, equivalently the category of antimatroids.

Let $U : \mathsf{DSC} \to \mathsf{FinSet}$ denote the forgetful functor, namely $(E, \text{dep}) \mapsto E$. Let us show that this functor has a left adjoint. Indeed, define the free functor $F : \mathsf{FinSet} \to \mathsf{DSC}$ by $F(S) = (S, \text{dep}_S)$, where $\text{dep}_S : S \to P(P(S))$ is defined by $\text{dep}_S(s) = \{\varnothing\}$. Namely $F(S)$ is the discrete DSC on S. Clearly if $f : S \to U(E, \text{dep})$ is a set function, then f will be a map of DSCs $f : F(S) \to (E, \text{dep})$, as $\varnothing \subseteq f^*(D^{f(s)} \cup s)$ for all $s \in S$. This proves the following result.

Lemma 5.1. The above functors define an adjunction $F : \mathsf{FinSet} \rightleftarrows \mathsf{DSC} : U$. Further, the forgetful functor $U : \mathsf{DSC} \to \mathsf{FinSet}$ is faithful, making DSC into a concrete category.

This result implies that U preserves whatever limits exist in DSC. Thus if a limit of a diagram of DSCs exists, it must be a DSC whose underlying set is a limit of the underlying sets in the diagram.

It is easy to see that DSC has both an initial and terminal object. Consider the empty set $\varnothing$ equipped with the unique function $\text{dep}_\varnothing : \varnothing \to P(P(\varnothing))$. This is vacuously a DSC. Clearly there is a unique morphism $(\varnothing, \text{dep}_\varnothing) \to (E, \text{dep})$ for any DSC. Similarly there is a unique map $(E, \text{dep}) \to F(*) = (*, \text{dep}_*)$, where $*$ denotes the singleton set. Thus we have proven the following result.

Proposition 5.2. The category DSC has initial and terminal objects.

Since $U : \mathsf{DSC} \to \mathsf{FinSet}$ preserves limits, if $f : (E, \text{dep}) \to (E', \text{dep}')$ is a monomorphism of DSCs, then it must be a monomorphism of sets. Now f is a monomorphism if and only if for every DSC X, the function $\mathsf{DSC}(X,f):\mathsf{DSC}(X,E)\to\mathsf{DSC}(X,E')$ is injective. For every $X\in\mathsf{DSC}$ we have the following commutative diagram

$$\begin{array}{ccc} \mathsf{DSC}(X,E) & \xrightarrow{\mathsf{DSC}(X,f)} & \mathsf{DSC}(X,E') \\ {\scriptstyle U_E}\downarrow & & \downarrow{\scriptstyle U_{E'}} \\ \mathsf{FinSet}(UX,UE) & \xrightarrow[\mathsf{FinSet}(UX,Uf)]{} & \mathsf{FinSet}(UX,UE') \end{array}$$

and since U is faithful, the vertical maps are injective. Thus if Uf is injective, then so is $\mathsf{FinSet}(UX,Uf)$, and therefore so is $\mathsf{DSC}(X,f)$. Thus we have established the following.

Lemma 5.3. A map $f:(E,\mathrm{dep})\to(E',\mathrm{dep}')$ of DSCs is a monomorphism if and only if Uf is an injective function.

Similarly f is an epimorphism if and only if for every DSC X, the function $\mathsf{DSC}(f,X):\mathsf{DSC}(E',X)\to\mathsf{DSC}(E,X)$ is injective. The same reasoning as above then proves the following result.

Lemma 5.4. If $f:E\to E'$ is a map of DSCs such that $U(f)$ is a surjection of sets, then f is an epimorphism of DSCs.

We now wish to show that if a morphism $f:(E,\mathrm{dep})\to(E',\mathrm{dep}')$ of DSCs is an epimorphism, then $U(f)$ is a surjection of sets. So suppose $f:E\to E'$ is a morphism of DSCs such that $U(f)$ is not a surjection. We wish to show that it cannot be an epimorphism in the category DSC. To do so, let us define a construction on DSCs that "doubles" an element.

Let (E,dep) be a DSC, and let $b\in E$. We will define a preDSC $E_{b\to b_1,b_2}$ as follows. Let the underlying set of $E_{b\to b_1,b_2}$ be defined as $(E\setminus b)\cup\{b_1,b_2\}$. Let us define its dependency structure dep' as follows. Let $\mathrm{dep}'(b_1)=\mathrm{dep}'(b_2)=\mathrm{dep}(b)$. If $e\in E$ such that none of e's depsets contain b, then let $\mathrm{dep}'(e)=\mathrm{dep}(e)$. If $e\in E$ such that it has at least one depset containing b, say $b\in D^e$ with $D^e\in\mathrm{dep}(e)$, then let $D^e_{b\to b_1,b_2}=(D^e\setminus b)\cup\{b_1,b_2\}$. This will still be a complete as a subset of $E_{b\to b_1,b_2}$ since $\mathrm{dep}'(b_1)=\mathrm{dep}'(b_2)=\mathrm{dep}(b)$ and D^e was complete in E. Let $\mathrm{dep}'(e)$ contain all the $D^e\in\mathrm{dep}(e)$ that don't contain b and all of the $D^e_{b\to b_1,b_2}$. This defines a preDSC, and it is easy to check that it is in fact a DSC.

Proposition 5.5. Let $f:(E,\mathrm{dep})\to(E',\mathrm{dep}')$ be a morphism of DSCs such that $U(f)$ is not a surjection. Then f is not an epimorphism in DSC.

Proof. If $U(f)$ is not a surjection, then there exists a $b\in E'$ such that $b\notin f(E)$. Let $E''=E'_{b\to b_1,b_2}$ be the "doubling" construction defined above, and consider two functions $g_1,g_2:E'\to E''$ with $g_1(e')=g_2(e')=e'$ for all $e'\neq b$, and have $g_1(b)=b_1$ and $g_2(b)=b_2$. It is easy to see from the construction that both g_1 and g_2 will be maps of DSCs that agree on $E'\setminus b$ but differ on b. Since $g_1\neq g_2$, this implies that f is not an epimorphism. □

Corollary 5.6. A morphism $f:(E,\mathrm{dep})\to(E',\mathrm{dep}')$ of DSCs is an epimorphism if and only if $U(f)$ is a surjection of sets.

Consider two DSCs $(E, \mathrm{dep}), (E', \mathrm{dep}')$. We can take the coproduct of their underlying sets $E + E'$, and consider the function $\mathrm{dep}_+ : E + E' \to P(P(E + E'))$ defined by

$$\mathrm{dep}_+(e) = \begin{cases} \mathrm{dep}(e) & \text{if } e \in E \\ \mathrm{dep}'(e) & \text{if } e \in E'. \end{cases}$$

It is not hard to show that $(E + E', \mathrm{dep}_+)$ is a DSC.

Lemma 5.7. Given two DSCs $(E, \mathrm{dep}), (E', \mathrm{dep}')$, the DSC $(E + E', \mathrm{dep}_+)$ is their categorical coproduct in the category DSC.

Proof. It is obvious that the inclusion maps $\mathrm{inl} : (E, \mathrm{dep}) \to (E + E', \mathrm{dep}_+)$ and $\mathrm{inr} : (E', \mathrm{dep}') \to (E + E', \mathrm{dep}_+)$ are maps of DSCs. So suppose that there are maps $f : (E, \mathrm{dep}) \to (Q, \mathrm{dep}^Q)$ and $g : (E', \mathrm{dep}') \to (Q, \mathrm{dep}^Q)$. We wish to show that the induced set function $(f, g) : E + E' \to Q$ is a map of DSCs. Namely we need to show that for every $x \in E + E'$, and every $D^q \in \mathrm{dep}^Q(q)$ where $q = (f, g)(x)$, there exists a $D^x \in \mathrm{dep}_+(x)$ such that $D^x \subseteq (f, g)^{-1}(D^q \cup q)$. WLOG suppose that $x = e$ for some $e \in E$, then $(f, g)(x) = f(e)$. Since f is a map of DSCs, this implies that there exists a $D^e \in \mathrm{dep}(e)$ such that $D^e \subseteq f^{-1}(D^q \cup q)$. Now since $f^{-1}(D^q \cup q)$ is a subset of E, we can also consider it as a subset of $E + E'$. In this sense it is clear that $f^{-1}(D^q \cup q) \subseteq (f, g)^{-1}(D^q \cup q)$, as the latter is equal to $f^{-1}(D^q \cup q) + g^{-1}(D^q \cup q)$. The case for $x = e'$ with $e' \in E'$ is similar. Thus (f, g) is a map of DSCs. It is not hard to check that (f, g) is the unique map of DSCs such that $(f, g) \circ \mathrm{inl} = f$ and $(f, g) \circ \mathrm{inr} = g$. □

This also proves that the forgetful functor $U : \mathsf{DSC} \to \mathsf{FinSet}$ preserves coproducts.

Let (E, dep) and (E', dep') be DSCs. Consider the set theoretic product $E \times E'$. We wish to put a DSC structure on this set. Let $(e, e') \in E \times E'$. Define a function $\mathrm{dep}_\times : E \times E' \to PP(E \times E')$ by $\mathrm{dep}_\times(e, e') = \{e \times D^{e'} \cup D^e \times e' \cup D^e \times D^{e'} : D^e \in \mathrm{dep}(e), D^{e'} \in \mathrm{dep}'(e')\}$. Namely a set $D^{(e,e')}$ is a possible dependency set of (e, e') if it is of the form $e \times D^{e'} \cup D^e \times e' \cup D^e \times D^{e'}$ for some possible dependency sets D^e and $D^{e'}$. It is not hard to check that $(E \times E', \mathrm{dep}_\times)$ is a DSC.

Proposition 5.8. Given two DSCs $(E, \mathrm{dep}), (E', \mathrm{dep}')$, the pair $(E \times E', \mathrm{dep}_\times)$ is a DSC, and is the categorical product of (E, dep) and (E', dep') in the category DSC.

Proof. First we must show that the projection maps $\pi_1 : (E \times E', \mathrm{dep}_\times) \to (E, \mathrm{dep})$ and $\pi_2 : (E \times E', \mathrm{dep}_\times) \to (E', \mathrm{dep}')$ are maps. We will show this for π_1. If $(e, e') \in E \times E'$, then we need to show that for every $D^e \in \mathrm{dep}(e)$, there exists a $D^{(e,e')} \in \mathrm{dep}(e, e')$ such that

$$D^{(e,e')} \subseteq \pi_1^{-1}(D^e \cup e). \tag{7}$$

Now $\pi_1^{-1}(D^e \cup e) = D^e \times E' \cup e \times E'$. Thus for any $D^{e'} \in \mathrm{dep}'(e')$, we have $e \times D^{e'} \cup D^e \times e' \cup D^e \times D^{e'} \subseteq e \times E' \cup D^e \times E'$. Thus (7) holds. The case for π_2 holds similarly.

Now let us prove that $(E \times E', \mathrm{dep}_\times)$ satisfies the universal property of a product. Namely if we have maps of DSCs $f : (Q, \mathrm{dep}^Q) \to (E, \mathrm{dep})$ and $g : (Q, \mathrm{dep}^Q) \to (E', \mathrm{dep}')$, we wish to show that the set theoretic map $\langle f, g \rangle : Q \to E \times E'$ is a map of DSCs. Namely if $q \in Q$, $f(q) = e, g(q) = e'$ and $D^{(e,e')} \in \mathrm{dep}_\times(e, e')$, then there exists a $D^q \in \mathrm{dep}^Q(q)$ such that

$$D^q \subseteq \langle f, g \rangle^{-1}\left(D^{(e,e')} \cup (e, e')\right). \tag{8}$$

Now $D^{(e,e')} = D^e \times D^{e'} \cup D^e \times e' \cup e \times D^{e'}$ for some $D^e \in \text{dep}(e), D^{e'} \in \text{dep}'(e')$, and thus

$$\langle f,g\rangle^{-1}\left(D^{(e,e')} \cup (e,e')\right) = [f^{-1}(D^e) \cap g^{-1}(D^{e'})] \cup [f^{-1}(D^e) \cap g^{-1}(e')] \cup [f^{-1}(e) \cap g^{-1}(D^{e'})] \cup [f^{-1}(e) \cap g^{-1}(e')] \tag{9}$$

Now since f and g are maps, we have $D^q \subseteq f^{-1}(D^e) \cup f^{-1}(e)$ and $D^q \subseteq g^{-1}(D^{e'}) \cup g^{-1}(e')$. Thus $D^q \subseteq (f^{-1}(D^e) \cup f^{-1}(e)) \cap (g^{-1}(D^{e'}) \cup g^{-1}(e'))$. Now if we let

$$f^{-1}(D^e) = X,\ g^{-1}(e') = Y,\ g^{-1}(D^{e'}) = Z,\ f^{-1}(e) = W,$$

then we have

$$\begin{aligned}(f^{-1}(D^e) \cup f^{-1}(e)) \cap (g^{-1}(D^{e'}) \cup g^{-1}(e')) &= (X \cup W) \cap (Z \cup Y)\\ &= (X \cap (Z \cup Y)) \cup (W \cap (Z \cup Y))\\ &= (X \cap Z) \cup (X \cap Y) \cup (W \cap Z) \cup (W \cap Y).\end{aligned}$$

and this last expression is equal to

$$[f^{-1}(D^e) \cap g^{-1}(D^{e'})] \cup [f^{-1}(D^e) \cap g^{-1}(e')] \cup [f^{-1}(e) \cap g^{-1}(D^{e'})] \cup [f^{-1}(e) \cap g^{-1}(e')].$$

Which is precisely (9). Thus the map $\langle f,g\rangle$ is a map of DSCs. □

It is also easy to see that products of discrete DSCs will be discrete DSCs. In other words, the functor $F : \mathsf{FinSet} \to \mathsf{DSC}$ preserves products.

Let (E, dep) be a preDSC. We say that a depset $X \in \text{dep}(e)$ is a **minimal depset** if for every $Y \in \text{dep}(e)$ if $Y \subseteq X$, then $X = Y$.

Given a preDSC (E, dep) let $(E, \widetilde{\text{dep}})$ denote the preDSC where $\widetilde{\text{dep}}(e)$ is the set of minimal depsets of e. Then $(E, \widetilde{\text{dep}})$ is irredundant. We call $(E, \widetilde{\text{dep}})$ the **irredundant hull** of (E, dep).

Suppose that (E, dep) is a DSC, and $A \subseteq E$ is a subset. Let $\text{dep}|_A$ denote the function $\text{dep}|_A : A \to PP(A)$ defined by saying that $X \in \text{dep}|_A(a)$ if there exists a $D^a \in \text{dep}(a)$ such that $X = D^a \cap A$. The pair $(A, \text{dep}|_A)$ forms a preDSC, but in general it will not be an irredundant preDSC. So we can take its irredundant hull $(A, \widetilde{\text{dep}|_A})$. This will be an irredundant preDSC.

Lemma 5.9. If (E, dep) is a DSC, and $A \subseteq E$ is a subset, then $(A, \widetilde{\text{dep}|_A})$ is a DSC. Further, the inclusion $i : (A, \widetilde{\text{dep}|_A}) \to (E, \text{dep})$ is a map of DSCs.

Proof. Clearly $(A, \widetilde{\text{dep}|_A})$ satisfies (D0) by construction. Clearly (D1) and (D2) hold, since they hold for (E, dep). Let us show that it satisfies (D3), namely that every depset is complete. Suppose that $a \in A$, $D^a \in \widetilde{\text{dep}|_A}(a)$ and $x \in D^a$. Since $D^a = D_0^a \cap A$ for some $D_0^a \in \text{dep}(a)$, this implies that $x \in A$ and $x \in D_0^a$. Since (E, dep) is a DSC, this means that there exists some $D^x \in \text{dep}(x)$ such that $D^x \subseteq D_0^a$. Thus $D^x \cap A \subseteq D^a$. If $D^x \cap A$ is a minimal dependency set for x in $(A, \text{dep}|_A)$, then $D^x \cap A \in \widetilde{\text{dep}|_A}(x)$. If $D^x \cap A$ is not a minimal dependency set for x in $(A, \text{dep}|_A)$, then there exists some $D_0^x \in \text{dep}(x)$ such that $D_0^x \cap A$ is a minimal depset of x and $D_0^x \cap A \subseteq D^x \cap A$. Thus $D_0^x \cap A \subseteq D^a$ and $D_0^x \cap A \in \widetilde{\text{dep}|_A}(x)$. Since x was arbitrary D^a is complete. Thus $(A, \widetilde{\text{dep}|_A})$ satisfies (D3), and therefore is a DSC.

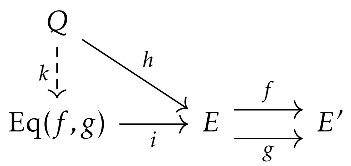
Let us show that the inclusion is a map of DSCs. Namely if $a \in A$, then for every $D^a \in \mathrm{dep}(a)$, there exists a $D_0^a \in \widetilde{\mathrm{dep}|_A}(a)$ such that $D_0^a \subseteq D^a \cup a$. But this is clearly true as one could take D_0^a to be the minimal depset contained in $D^a \cap A$. □

We say that $(A, \widetilde{\mathrm{dep}|_A})$ is the set A equipped with the subset DSC structure.

Now suppose that there are two DSC maps $f, g : (E, \mathrm{dep}) \to (E', \mathrm{dep}')$. We can consider the subset $\mathrm{Eq}(f,g) = \{e \in E : f(e) = g(e)\} \subseteq (E, \mathrm{dep})$, which is the categorical equalizer in FinSet. Thanks to Lemma 5.9, equipping $\mathrm{Eq}(f,g)$ with the subset DSC structure gives a DSC for which the inclusion map is a map of DSCs. We wish to show that this is a categorical equalizer of f and g.

Proposition 5.10. Given two maps of DSCs $f, g : (E, \mathrm{dep}) \to (E', \mathrm{dep}')$, the set $\mathrm{Eq}(f,g)$ equipped with the subset DSC structure from (E, dep), denoted by $(\mathrm{Eq}(f,g), \widetilde{\mathrm{dep}})$ is an equalizer for the two maps in the category DSC.

Proof. Suppose there is a DSC (Q, dep^Q) and a map $h : (Q, \mathrm{dep}^Q) \to (E, \mathrm{dep})$ such that $fh = gh$. We wish to show that h factors uniquely through the inclusion map $i : (\mathrm{Eq}(f,g), \widetilde{\mathrm{dep}}) \to (E, \mathrm{dep})$. We know that as set functions, h will factor uniquely through i, resulting in the following commutative diagram in FinSet.



we need only show that k is a map of DSCs. Namely if $q \in Q$ we wish to show that for every $D^{k(e)} \in \widetilde{\mathrm{dep}}(k(e))$ there exists a $D^q \in \mathrm{dep}^Q(q)$ such that $D^q \subseteq k^{-1}(D^{k(e)} \cup k(e))$. Since $h(Q) \subseteq \mathrm{Eq}(f,g)$, k is actually the corestriction of h, namely it is the same function with codomain restricted to $\mathrm{Eq}(f,g)$. Thus $D^{k(e)} = D^{h(e)}$, and this is of the form $D_0^{h(e)} \cap \mathrm{Eq}(f,g)$ for some $D_0^{h(e)} \in \mathrm{dep}(h(e))$ that makes $D^{h(e)}$ a minimal dependency set. Thus we need to show that there exists some D^q such that

$$D^q \subseteq h^{-1}\Big((D_0^{h(e)} \cap \mathrm{Eq}(f,g)) \cup h(e)\Big).$$

But the right hand side is equal to $h^{-1}(D_0^{h(e)}) \cap h^{-1}(\mathrm{Eq}(f,g)) \cup h^{-1}(h(e))$. Since h is equalized by f and g, $h^{-1}(\mathrm{Eq}(f,g)) = Q$. Thus we need only show that there exists a D^q such that $D^q \subseteq h^{-1}(D_0^{h(e)} \cup h(e))$. This is guaranteed since h is a map of DSCs. Thus k is a map of DSCs, and therefore an equalizer in DSC. □

Corollary 5.11. The category DSC is finitely complete.

Proof. The category DSC has a terminal object by Proposition 5.2, binary products by Proposition 5.8 and equalizers by Proposition 5.10. Since every finite limit can be written using a terminal object, binary products and equalizers, DSC is finitely complete. □

The case for colimits of DSCs is more subtle than limits. The following example showcases this subtlety.

Example 5.12. Consider the DSC $a.b \vee c$, and the two maps $a, c : * \to a.b \vee c$. We wish to show that there does not exist a coequalizer of these two maps in the category of DSCs. Suppose there was, denote it by

$$* \overset{a}{\underset{c}{\rightrightarrows}} a.b \vee c \xrightarrow{f} E$$

Now since f is a coequalizer, we know that it must be an epimorphism. By Corollary 5.6, this implies that f is a surjection on the underlying sets. Since f is a coequalizer, we know that its image $f(a.b \vee c)$ must have cardinality ≤ 2, since f must identify a and c. This implies that $|E| \leq 2$, where $|E|$ denotes the cardinality of E. Up to isomorphism, there are exactly three DSCs with cardinality ≤ 2. They are the terminal DSC $*$, the discrete DSC on two elements x, y and the DSC $u.v$. Now there exists no map of DSCs $a.b \vee c \to x, y$, so it cannot be the coequalizer. There exists a map of DSCs $f : a.b \vee c \to u.v$ with $f(a) = f(c) = u$ and $f(b) = v$, and since this map isn't constant, it doesn't factor through $*$, so $*$ cannot be the coequalizer. Thus $u.v$ is the only possible candidate for E. However, consider the map $h : a.b \vee c \to r.s \vee t$ with $h(a) = h(c) = r$ and $h(b) = s$. This is a map of DSCs, and if $u.v$ was the coequalizer, then h would have to factor uniquely through f, namely there would have to exist a unique map of DSCs $k : u.v \to r.s \vee t$ such that $kf = h$. This implies that $k(u) = r$ and $k(v) = s$. However this is not a map of DSCs. For instance $k^*(t \cup r) = u$, so there exists no depset of u in $u.v$ that is a subset of $k^*(t \cup r)$. Thus f doesn't factor uniquely through h, thus $u.v$ cannot be the coequalizer. Thus there cannot exist any coequalizer.

Corollary 5.13. The category DSC does not have all coequalizers.

However by Corollary 4.9, we know that the full subcategory of DSNCs is finitely cocomplete, because the category of finite posets FinPos is. We give an example of a coequalizer computation, which could have also been computed in the category FinPos.

Example 5.14. Consider the DSC with $E = \{a, b, c\}$ and $\text{dep}(a) = \{\{b, c\}\}, \text{dep}(b) = \{\{c\}\}$, and $\text{dep}(c) = \{\varnothing\}$. Consider the maps $* \rightrightarrows (E, \text{dep})$ of DSCs that pick out the points a and c, and where $*$ denotes the terminal DSC. We claim that the coequalizer of these morphisms exists in DSC and is the terminal DSC. Indeed, suppose that $f : (E, \text{dep}) \to (E', \text{dep}')$ was a map of DSCs that coequalized the maps a, c. Then $f(a) = f(c)$, and $\{b, c\} \subseteq f^{-1}(D^{f(a)} \cup f(a))$ for every $D^{f(a)} \in \text{dep}'(f(a))$. Suppose that $f(a) \neq f(b)$. Then $f^{-1}(f(a)) = \{a, c\}$, so $f(b) \in D^{f(a)}$. But we also have that $\{c\} \subseteq f^{-1}(D^{f(b)} \cup f(b))$ for every depset $D^{f(b)}$, so that $f(c) = f(a) \in D^{f(b)}$ since $f^{-1}(f(b)) = b$. But E' is a DSC, so this implies that $f(b) \in D^{f(b)}$ for every depset $D^{f(b)}$, which is a contradiction. Thus $f(a) = f(b) = f(c)$. From this it follows that the coequalizer of the maps $a, c : * \rightrightarrows (E, \text{dep})$ is the terminal DSC.

6. The Bruns-Lakser Completion

In this section we will consider a construction known in the literature as the **Bruns-Lakser Completion** or **Bruns-Lakser Injective Envelope**. It was defined in [BL70], and it defines a idempotent way of completing a meet-semilattice to a distributive lattice in a way that preserves certain kinds of joins. Here we will describe this construction, apply it to the image of lattices under rdp and show how the composition of Bruns-Lakser after rdp provides an interesting interpretation of DSCs as a kind of data structure known as a Merkle tree.

First we will recall the classical Bruns-Lakser completion. Later we will recast this construction as the Yoneda embedding of a meet-semilattice into its category of "thin sheaves," as described in [Stu05].

Definition 6.1. Let L be a meet-semilattice. We say that a subset $S \subseteq L$ is a **distributive subset** if for every $x \in L$, the following joins exist and the following identity holds:

$$x \wedge \bigvee S = \bigvee (x \wedge S),$$

where $\bigvee(x \wedge S)$ is the join of the subset $\{x \wedge s : s \in S\}$. We call elements of the form $\bigvee S$ for a distributive subset a **distributive join**. We say a map $f : L \to L'$ of meet-semilattices preserves distributive joins if whenever S is a distributive subset of L, then $f_*(S)$ is a distributive subset of L', and $f(\bigvee S) = \bigvee f_*(S)$.

Remark 6.2. In some papers such as [GG13] and [BL70] they refer to distributive subsets as admissible subsets. We use the name distributive subset as it is more closely related to its definition.

We will soon see that distributive joins are in some sense canonical to a meet-semilattice. For us, the condition on the joins existing is not of concern, since $\mathsf{rdp}(E)$ has joins for any DSC E. However, not all subsets will be distributive, and thus even though the Bruns-Lakser construction was originally defined for meet-semilattices, it will apply to our case as well.

Definition 6.3. A **distributive ideal** in a meet-semilattice L is a subset $A \subseteq L$ such that:

(1) if $x \in A$ and $y \leq x$, then $y \in A$, and
(2) if $I \subseteq A$ and I is a distributive subset of L, then $\bigvee I \in A$.

Let $\mathsf{BL}(L)$ denote the subposet of $P(L)$ of distributive ideals ordered under set inclusion. We refer to this as the **Bruns-Lakser completion** of L.

Lemma 6.4. If $a \in L$, then $\downarrow a$ is a distributive ideal.

Proof. If $I \subseteq \downarrow a$ is any subset such that $\vee I$ exists, then $\vee I \in \downarrow a$. □

Let $\mathbf{b} : L \to \mathsf{BL}(L)$ denote the injective order-preserving map $a \mapsto (\downarrow a)$. We wish to show that this map preserves meets and distributive joins, so first we need to describe that structure in $\mathsf{BL}(L)$.

Lemma 6.5 ([BL70, Lemma 3])**.** For a meet-semilattice L, $\mathsf{BL}(L)$ is a complete lattice, namely it has infinite joins and meets. Further, meets correspond to intersection, while joins are given by:

$$\bigvee_{i \in I} A_i = \left\{ \bigvee X : X \subseteq \bigcup_{i \in I} A_i,\ X \text{ is distributive in } L \right\},$$

where $\{A_i\}_{i \in I}$ is a collection of elements of $\mathsf{BL}(L)$.

Corollary 6.6. The function $\mathbf{b} : L \to \mathsf{BL}(L)$ preserves meets and distributive joins.

Lemma 6.7 ([BL70, Corollary 1])**.** For any meet-semilattice L, $\mathsf{BL}(L)$ is a complete infinitely distributive lattice.

Proof. We need to show that if X and $\{A_i\}_{i\in I}$ belong to $\mathsf{BL}(L)$, then

$$X \wedge \bigvee_{i\in I} A_i = \bigvee_{i\in I} (X \wedge A_i).$$

But the left hand side is the set

$$X \cap \left\{ \bigvee Y \,:\, Y \subseteq \bigcup A_i,\ Y \text{ distributive.} \right\}$$

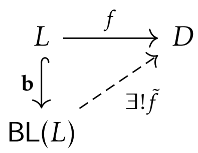
which is equal to the set

$$\left\{ \bigvee Y \,:\, Y \subseteq \bigcup (X \cap A_i),\ Y \text{ distributive.} \right\}.$$

This is precisely the right hand side. □

In fact, the Bruns-Lakser completion satisfies the following universal property:

Theorem 6.8 ([BL70], [GG13])**.** If D is a distributive lattice, and $f : L \to D$ preserves arbitrary meets and joins of distributive subsets, then there exists a unique map of distributive lattices $\tilde{f} : \mathsf{BL}(L) \to D$ making the following diagram commute:



We wish to view DSCs as some kind of generalized space, in the hope of being able to bring to bear modern mathematical tools to study package management systems. The particular generalization of space we will use is known as a **locale**. They are a generalization of the structure of the lattice of open subsets of a topological space. One can also think of the theory of locales as a sort of decategorified topos theory, see [Joh82]. This is the viewpoint we will take here.

Definition 6.9. Let L be a finite lattice. A **thin presheaf** on L is a functor (order-preserving map) $\varphi : L^{op} \to 2$, where 2 denotes the poset $2 := \{0 \leq 1\}$. A morphism of thin presheaves will be a natural transformation of such functors. Let $2\mathsf{Pre}(L)$ denote the category of thin presheaves on L.

As is usual in topos theory, we desire a notion of thin sheaf. This will come about through the notion of a Grothendieck pretopology on a lattice. Here there is no need for an auxiliary definition, the usual definition of Grothendieck pretopology suffices, but in the case of a lattice reduces to the following.

Definition 6.10. Let L be a finite lattice, and $x \in L$. We say that a subset $R \subseteq L$ is a **family** over x if whenever $r \in R$, then $r \leq x$.

Definition 6.11. A **Grothendieck pretopology** on a finite lattice L is a function J that assigns to every point $x \in L$ a set of families $J(x)$ satisfying the following conditions:

(1) The singleton $x \in J(x)$,
(2) if $R \in J(x)$ and $y \leq x$, then $R \wedge y \in J(y)$, and

(3) if $R \in J(x)$, and if for every $r_i \in R$, there exists a family $S_i \in J(r_i)$, then $\bigcup_i S_i \in J(x)$.

We call the families $R \in J(x)$ that belong to a Grothendieck pretopology a **covering family**. We call a lattice equipped with a Grothendieck pretopology a **posite**. A morphism of posites $f : L \to L'$ is an order preserving map that preserves meets and covering families, in the sense that if $R \in J(x)$, then $f_*(R) \in J(f(x))$.

Let L be a lattice, then define J_{dis} to be the function that assigns to every $x \in L$ the set of distributive families $R \subseteq L$ such that $\bigvee R = x$. We leave it to the reader to check that this defines a Grothendieck pretopology on L. In fact J_{dis} generates what is known as the **canonical Grothendieck topology** on L, see [Stu05, Theorem 1].

Let us now consider our example, $\mathsf{rdp}(E)$, where E is a DSC. This is a finite lattice, and thus it can be equipped with J_{dis}, and thus canonically is posite. A morphism of such posites is what is called a $\langle \wedge, a\vee \rangle$-morphism in [GG13]. Namely it is an order-preserving function that preserves meets, distributive joins, and the image of a distributive subset is a distributive subset. We wish to understand what kinds of morphsims of DSCs $f : E \to E'$ induce morphisms of posites $f^* : \mathsf{rdp}(E') \to \mathsf{rdp}(E)$ when equipped with J_{dis}.

Lemma 6.12. If $f : E \to E'$ is a distributive-preserving morphism as given in Definition 2.50, then $f^* : \mathsf{rdp}(E') \to \mathsf{rdp}(E)$ is a morphism of posites.

Proof. Since distributive-preserving morphisms are morphisms of DSCs, this means that $f^* : \mathsf{rdp}(E') \to \mathsf{rdp}(E)$ is a lattice map, and thus preserves all meets and joins. We need only check that it preserves distributive subsets, but the proof that f^* being surjective is sufficient to prove this is [GG13, Proposition 3.12]. □

Let $\mathsf{DSC}_{\text{dis}}$ denote the category of DSCs with distributive-preserving morphisms.

Definition 6.13. A thin presheaf $\varphi : L^{op} \to 2$ on a posite (L, J), is a **thin sheaf** if for all $x \in L$ and for all $R \in J(x)$ if $\varphi(r) = 1$ for all $r \in R$ then $\varphi(x) = 1$. Let $2\mathsf{Sh}(L)$ denote the full subcategory of thin presheaves on the thin sheaves over L.

For a general category C with a Grothendieck topology, known as a site, the category of sheaves $\mathsf{Sh}(\mathsf{C})$ is (by definition) a Grothendieck topos. The decategorified version of this statement is that for a posite (L, J), the category of thin sheaves $2\mathsf{Sh}(L)$ is a locale.

Definition 6.14. A **locale** is a poset with infinite joins, finite meets and satisfies an infinite version of the distributive law.

From the definition, it is obvious that finite locales are precisely finite distributive lattices. There is a canonical monomorphism $y : L \hookrightarrow 2\mathsf{Sh}(L)$ given by the decategorified Yoneda embedding, namely sending x to the map $yx : L^{op} \to 2$ defined by

$$yx(u) = \begin{cases} 1 & \text{if } u \leq x \\ 0 & \text{else.} \end{cases}$$

One can show that this embedding is universal in the same sense as the Bruns-Lakser completion:

Theorem 6.15 ([Stu05])**.** Let L be a meet-semilattice, then there is an isomorphism of distributive lattices

$$\mathsf{BL}(L) \cong 2\mathsf{Sh}(L).$$

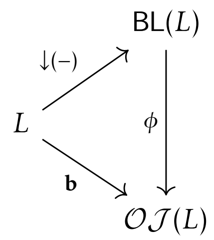
If $f : (E, \mathrm{dep}) \to (E', \mathrm{dep}')$ is a morphism in $\mathsf{DSC}_{\mathsf{dis}}$, then it induces a map $\mathsf{BL}(\mathsf{rdp}(E')) \to \mathsf{BL}(\mathsf{rdp}(E))$ as shown in [GG13, Corollary 3.13]. This shows that the Bruns-Lakser completion applied to rdp of a DSC defines a functor $\mathsf{BL} \circ \mathsf{rdp} : \mathsf{DSC}_{\mathsf{dis}} \to \mathsf{FinDLatt}^{op}$.

Now we give a third and final characterization of the Bruns-Lakser completion in the case we are interested in, namely a finite lattice. Recall the downset functor $\mathcal{O}$ from Definition 3.6.

Proposition 6.16. Let L be a finite lattice, then the map

$$\phi : \mathsf{BL}(L) \to \mathcal{O}\mathcal{J}(L),$$

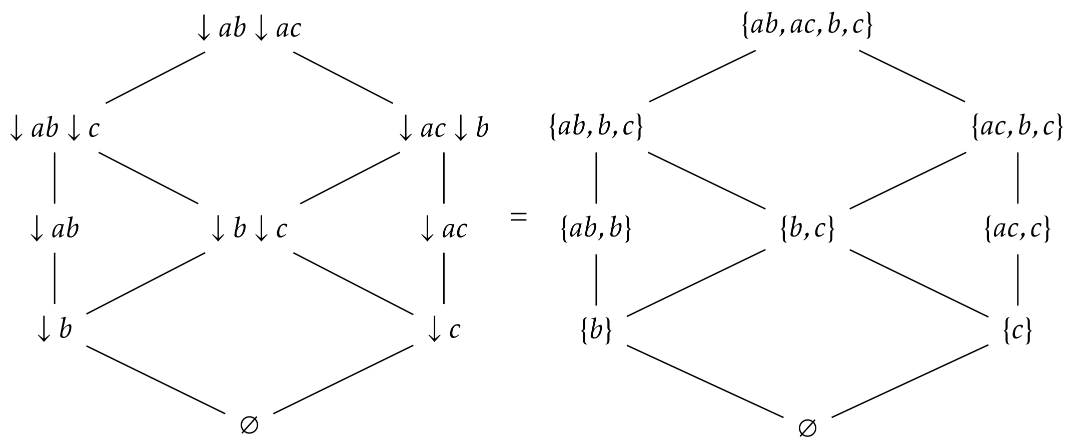
defined by $\phi(A) = \bigcup_{a \in A} \widehat{a}$, where $\widehat{a} = \{x \in \mathcal{J}(L) : x \leq a\}$ is an isomorphism of distributive lattices. Further it makes the following diagram commute:



Proof. This follows from [GG13, Proposition 3.5] as follows. In [GG13]'s notation, L^δ denotes a completion of L, but when L is a finite lattice, it is complete, so for this proof $L^\delta \cong L$. Now we need only show that $\mathcal{O}\mathcal{J}(L)$ is precisely the sublattice R of $P\mathcal{J}(L)$ generated by the downsets of the form $\widehat{x}$ for $x \in L$, where $P\mathcal{J}(L)$ is the lattice of all subsets of $\mathcal{J}(L)$. However if X is some downset of $\mathcal{J}(L)$, then clearly $X = \bigvee_{x \in X} \widehat{x} = \bigcup_{x \in X} \widehat{x}$ in $\mathcal{O}\mathcal{J}(L)$, so $\mathcal{O}\mathcal{J}(L) \subseteq R$. But $\mathcal{O}\mathcal{J}(L)$ also is a sublattice that contains all of the $\widehat{x}$, thus $R \subseteq \mathcal{O}\mathcal{J}(L)$. Thus they are equal, proving the Proposition. □

Example 6.17. Thanks to Proposition 6.16, it is easy to compute examples of the Bruns-Lakser completion. Let us consider what $\mathsf{BL}(L)$ looks like when $L = \mathsf{rdp}(a.b \vee c)$ from Example 2.18. The join irreducibles of $\mathsf{rdp}(a.b \vee c)$ are $\mathcal{J}(L) = \{b, c, ab, ac\}$. Thus $\mathcal{J}(L)$ looks like

Now if we let $\downarrow b, \downarrow c, \downarrow ab, \downarrow ac$ all denote their respective downsets in $\mathcal{J}(L)$, and denote unions by concatenation, i.e. $\downarrow b \cup \downarrow c = \downarrow b \downarrow c$, then $\mathsf{BL}(L)$ looks like:



One can see easily how L embeds into $\mathsf{BL}(L)$ by

$$b \mapsto \downarrow b, \qquad c \mapsto \downarrow c, \qquad ab \mapsto \downarrow ab, \qquad ac \mapsto \downarrow ac$$
$$bc \mapsto \downarrow b \downarrow c, \qquad abc \mapsto \downarrow ab \downarrow ac.$$

Given a DSC (E, dep), we can consider the DSNC $\tau(\mathsf{BL}(\mathsf{rdp}(E)))$, where τ denotes the inverse of the equivalence of Corollary 4.10. This defines a composite functor $(\tau \circ \mathsf{BL} \circ \mathsf{rdp}) : \mathsf{DSC}_{\mathsf{dis}} \to \mathsf{DSNC}$. Conceptually, whenever an event has multiple possible dependency sets, this composite functor splits such events into multiple distinct events, each with one dependency set.

In fact the Bruns-Lakser completion of $\mathsf{rdp}(E)$ has a very interesting interpretation in terms of Merkle trees. Merkle trees were first introduced as a mechanism for digital signatures [Mer87] but now are used widely from storage systems to source control to the hash structure of packages in the Nix package management system. The main idea of the last of these is that it stores "software components in isolation from each other in a central component store, under path names that contain cryptographic hashes of all inputs involved in building the component" [Dol06]. This establishes, effectively, a tree, where each node can be thought of depending on all nodes below it, and is labeled by its own data combined with (the recursive hash of) the subtree of all nodes below it. The act of distinguishing multiple copies of "the same" event by the record of the choices of their possible inputs corresponds to the construction of a component store in Nix. There is thus a sense in which action of the Bruns-Lakser completion on DSCs yields "the Merkle function".

7. Version Parametrization

In keeping with our intended notion of DSCs as presenting collections of packages, this section attempts to give some meaningful semantics to the informal notion of a "version". Typically, a version is just some monotone increasing identifier attached to a computer program or software package with no special meaning. However, there are often things such as package version policies, or semantic versioning, which attempt to give some "subtype"-like meaning to versioning. In particular, the idea is that to upgrade from a lower version to a higher version should make things in a sense better, rather than worse – i.e., provide new features, fix bugs, etc. In such version policies, these universally better, non-breaking updates to a package are known as minor version releases – as opposed to major version releases, which may change the functionality arbitrarily, including perhaps scrapping all the existing features for new ones intended to be used for a similar purpose.

Here we introduce a notion of a version relation that attempts to capture the idea of such a minor version update, i.e. when one event is universally more useful than another event. That is to say, when some event is enabled in at least the situations where the lower version is, and further, enables at least all the same events enabled by the lower version.

Definition 7.1. Let (E, dep) be a DSC. We introduce a relation on E, written $e \blacktriangleleft e'$ if

(1) $\mathrm{dep}(e) \subseteq \mathrm{dep}(e')$, and
(2) $D^x \in \mathrm{dep}(x)$ is a depset such that $e \in D^x$, then $(D^x \setminus e) \cup \{e'\} \in \mathrm{dep}(x)$ is also a depset.

We call this the **higher version relation**. We say that e' is a higher version of e.

Lemma 7.2. The higher version relation $\blacktriangleleft$ is reflexive and transitive.

Proof. It is clear that $\blacktriangleleft$ is reflexive. Now suppose that $e \blacktriangleleft e'$ and $e' \blacktriangleleft e''$. Then if $D^x \in \mathrm{dep}(x)$ for some $x \in E$ with $e \in X$, then $D_0^x = (D^x \setminus e) \cup \{e'\} \in \mathrm{dep}(x)$. But since $e' \blacktriangleleft e''$, and $e' \in D_0^x$, then $(D_0^x \setminus \{e'\}) \cup \{e''\} = (D^x \setminus e) \cup \{e''\} \in \mathrm{dep}(x)$. Thus $e \blacktriangleleft e''$. □

Given $e \in E$, let $\mathrm{Vers}(e)$ denote the set of $e' \in E$ such that $e \blacktriangleleft e'$. We call this the set of **higher versions** of e.

Now if (E, dep) is a DSC, then consider the following function, $V : P(E) \to P(E)$ defined by

$$V(X) = \bigcup_{x \in X} \mathrm{Vers}(x)$$

Lemma 7.3. If $X \subseteq E$ is a reachable subset, then $V(X)$ is a reachable subset.

Proof. Suppose that $e' \in V(X)$. Then there exists some $e \in X$ such that $e \blacktriangleleft e'$. Then $\mathrm{dep}(e) \subseteq \mathrm{dep}(e')$. Since X is complete, there exists some $D^e \subseteq X$. But $D^e \in \mathrm{dep}(e')$ and $D^e \subseteq X \subseteq V(X)$, thus $V(X)$ is complete and therefore reachable. □

Definition 7.4. Let P be a poset and $f : P \to P$ a function. We say that f is a **closure operator** if:

(1) f is order preserving,
(2) if $X \in P$, then $X \leq f(X)$, and
(3) if $X \in P$, then $f(f(X)) = f(X)$.

Note that this is precisely the same thing as an **idempotent monad** on P thought of as a category.

Corollary 7.5. Given a DSC (E, dep), V descends to a function $V : \mathsf{rdp}(E) \to \mathsf{rdp}(E)$, and is a closure operator.

Now we wish to describe how V descends to a closure operator on the Bruns-Lakser completion.

Lemma 7.6. Let L denote a finite lattice, and suppose that $S \in \mathcal{O}\mathcal{J}(L)$. Then

$$S = \bigcup_{x \in S} \mathbf{b}(x).$$

Proof. Suppose that $x \in S$. Then x is join irreducible, so $x \in \mathbf{b}(x)$, thus $x \in \bigcup_{x \in S} \mathbf{b}(x)$ and so $S \subseteq \bigcup_{x \in S} \mathbf{b}(x)$.

Now suppose that $y \in \bigcup_{x \in S} \mathbf{b}(x)$. Then $y \in \mathbf{b}(x)$ for some $x \in S$. Thus $y \leq x$ and y is join irreducible. But S is a down-closed subset of $\mathcal{J}(L)$, so $y \in S$, and therefore $\bigcup_{x \in S} \mathbf{b}(x) \subseteq S$. □

Now if $V : L \to L$ is a closure operator on a finite lattice, we want to show that it induces a closure operator $\mathbf{V} : \mathsf{BL}(L) \to \mathsf{BL}(L)$. If $S \in \mathsf{BL}(L) \cong \mathcal{O}\mathcal{J}(L)$, then by 7.6, we can write S as $S = \bigcup_{x \in S} \mathbf{b}(x)$. Thus define

$$\mathbf{V}(S) = \bigcup_{x \in S} \mathbf{b}(V(x)).$$

Proposition 7.7. The function $\mathbf{V} : \mathsf{BL}(L) \to \mathsf{BL}(L)$ as defined above is a closure operator.

Proof. Suppose that $S \subseteq T$. Then clearly $\mathbf{V}(S) \subseteq \mathbf{V}(T)$. Thus $\mathbf{V}$ is order preserving.

We wish to show that $S \subseteq \mathbf{V}(S)$. Suppose that $x \in S$. Then $x \leq V(x)$, since V is a closure operator on L. Thus $\mathbf{b}(x) \subseteq \mathbf{b}(V(x))$. Therefore $S \subseteq \bigcup_{x \in S} \mathbf{b}(V(x)) = \mathbf{V}(S)$.

Now we wish to show that $\mathbf{V}(\mathbf{V}(S)) \subseteq \mathbf{V}(S)$. Suppose that $z \in \mathbf{V}(\mathbf{V}(S))$. Then $z \in \mathbf{b}(V(y))$ for some $y \in \mathbf{V}(S)$. But then $y \in \mathbf{b}(V(x))$ for some $x \in S$. Therefore $y \leq V(x)$, so $V(y) \leq V(V(x)) \leq V(x)$, since V is a closure operator. Thus $\mathbf{b}(V(y)) \subseteq \mathbf{b}(V(x))$. Thus $z \in \mathbf{b}(V(x)) \subseteq \mathbf{V}(S)$. Thus $\mathbf{V}$ defines a closure operator. □

Thanks to Proposition 7.7, the closure operator $\mathbf{V} : \mathsf{BL}(\mathsf{rdp}(E)) \to \mathsf{BL}(\mathsf{rdp}(E))$ is now defined for any DSC (E, dep). We call this the **version monad** on $\mathsf{BL}(\mathsf{rdp}(E))$.

8. Conclusion

From a mathematical standpoint, this paper has made the following contributions. Theorem 3.15 shows that antimatroids can be used to model package management systems. By introducing DSCs we have provided a connection between the theory of general event structures and that of antimatroids. We have also proposed a natural definition of morphisms of antimatroids, and explored properties of the resulting category of antimatroids. Furthermore, we have given and shown applications of a simple characterization of the finite Bruns-Lakser completion.

From an applied standpoint, we have put this mathematics to work in providing a formal characterization of the dependency data of a package repository, as well as a formal characterization of "minor" versions of packages, and of package versioning policies. We have also given a mathematical account of the logical and order-theoretic interpretation of Merkle trees, which sheds light on the reasons for their widespread utility.

It is worth noting that the construction and characterization of Bruns-Lakser given in Proposition 6.16 in fact extends beyond finite lattices to all finite posets, as explored in [BP20].

There remain some interesting outstanding questions regarding morphisms of DSCs. It was established in Corollary 2.43 that the image of of morphisms of DSCs under rdp is a map of join-semilattices. It is an open question if this is a bi-implication. Similarly, it is open if Lemma 2.47 is a bi-implication, and if not, what an appropriate collection of morphisms would be to generate an equivalence of categories between some subcategory of DSC and finite lattices. Finally, it is an open question to characterize a larger class of maps of DSCs that makes BL functorial.

There are a number of possible directions for future work. One is further development of a category whose internal logic can encode package dependency problems. Another is modeling updates to package repositories by means of morphisms in the category of DSCs. Still another is extending DSCs with a notion of conflict, so that they would more fully correspond to general event structures. Beyond that is development of insights into efficient dependency solvers – a topic of great practical interest. Finally, there is much we hope that can be extracted from the surprising connection between dependency systems and antimatroids. There are many high-powered tools available for the study of the latter, as well as their cousins, matroids, and the more general class of greedoids to which both belong, not least characteristic polynomials. It would be of great interest to see what insights these tools can bring when transported to the world of dependency structures.

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